Test #4 Solution

Date: 04/23/2013 Name: ______________________

NOTE: You must show all work to earn credit.

1. (15 points) Find the area of the region bounded by the graphs of the equations:

$$xy = 9, \quad y = x, \quad y = 0, \quad x = 9.$$

Solution:

First, notice that $xy = 9$ and $y = x$ intersect at $(3, 3)$, which separates the region into two parts:

$$0 \le x \le 3, 0 \le y \le x, \text{and } 3 \le x \le 9, 0 \le y \le 9/x.$$

Next, we need to set up the iterated integral to evaluate the area:

$$A = \int_0^3 \int_0^x dy\, dx + \int_3^9 \int_0^{\frac{9}{x}} dy\, dx = \int_0^3 x dx + \int_3^9 \frac{9}{x} dx = \frac{9}{2} + 9 \ln 3$$

2. (15 points) Find the volume of the solid region bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the xy-plane. (Hint: $\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta = \frac{3\pi}{16}$)

Solution: (see Example 3 on page 997)

First, let $z = 0$ to determine the region in the xy-plane bounded by $x^2 + 2y^2 = 4$. So,

$$-2 \le x \le 2, \qquad -\sqrt{\frac{4-x^2}{2}} \le y \le \sqrt{\frac{4-x^2}{2}}$$

The volume is

$$V = \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (4 - x^2 - 2y^2) dy\, dx = \int_{-2}^{2} \left((4-x^2)y - \frac{2}{3}y^3 \right) \Big]_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$= \int_{-2}^{2} \left(2(4-x^2)\sqrt{\frac{4-x^2}{2}} - \frac{4}{3}\left(\frac{4-x^2}{2}\right)\sqrt{\frac{4-x^2}{2}} \right) dx$$

$$= \int_{-2}^{2} \left[\frac{2}{\sqrt{2}}(4-x^2)\sqrt{4-x^2} - \frac{2}{3\sqrt{2}}(4-x^2)\sqrt{4-x^2} \right] dx$$

$$= \int_{-2}^{2} \left[\frac{4}{3\sqrt{2}}(4-x^2)\sqrt{4-x^2} \right] dx = \frac{4}{3\sqrt{2}} \int_{-2}^{2} (4-x^2)^{\frac{3}{2}} dx \qquad \text{let } x = 2\sin\theta$$

$$= \frac{4}{3\sqrt{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 16\cos^4\theta\, d\theta = \frac{64}{3\sqrt{2}}(2) \int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta = \frac{128}{3\sqrt{2}}\left(\frac{3\pi}{16}\right) \qquad \text{Wallis' theorm}$$

$$= 4\sqrt{2}\pi$$

3. (15 points) Evaluate the iterated integral by switching the order of integration.

$$\int_0^2 \int_{\frac{1}{2}x^2}^2 \sqrt{y}\cos y\,dy\,dx$$

Solution:

First, notice that the iterated integral implies that the region is vertically simple. However, the region is also horizontally simple. We can change

$$0 \le x \le 2, \qquad \frac{1}{2}x^2 \le y \le 2$$

to

$$0 \le y \le 2, \qquad 0 \le x \le \sqrt{2y}$$

Therefore the original iterated integral can be changed to

$$\int_0^2 \int_{\frac{1}{2}x^2}^2 \sqrt{y}\cos y\,dy\,dx = \int_0^2 \int_0^{\sqrt{2y}} \sqrt{y}\cos y\,dx\,dy = \int_0^2 \sqrt{y}\cos y\sqrt{2y}\,dy$$

$$= \sqrt{2}\int_0^2 y\cos y\,dy = \sqrt{2}(y\sin y + \cos y)\Big]_0^2 = \sqrt{2}(2\sin 2 + \cos 2 - 1)$$

4. (15 points) Use polar coordinates to set up and evaluate the double integral $\iint_R f(x,y)\,dA$:

$$f(x,y) = e^{-(x^2+y^2)/2}, \qquad R: x^2 + y^2 \le 25, x \ge 0.$$

Solution:

In polar coordinate system, we have

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad dA = r\,dr\,d\theta$$

Notice that $x \ge 0$, the region can be represented in the polar coordinates as

$$R: -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, 0 \le r \le 5.$$

Therefore,

$$\iint_R f(x,y)dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^5 e^{\frac{-r^2}{2}} r dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(-e^{\frac{-r^2}{2}}\right)\Big]_0^5 d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 - e^{-\frac{25}{2}}\right) d\theta = \pi\left(1 - e^{-25/2}\right)$$

5. (15 points) Find the mass and center of mass of the lamina bounded by the graphs of the equations for the given density.

$$y = 9 - x^2, \qquad y = 0, \qquad \rho = ky^2.$$

Solution:

When $y = 0$, $x = \pm 3$. So the region is $R\colon -3 \le x \le 3, 0 \le y \le 9 - x^2$.

We need to find m, M_x, and M_y.

$$m = \int_{-3}^{3}\int_{0}^{9-x^2} ky^2 dy\, dx = \int_{-3}^{3} \frac{k}{3}(9-x^2)^3 dx = \frac{2k}{3}\int_{0}^{3}(9-x^2)^3 dx$$

$$= \frac{2k}{3}\int_{0}^{3}(729 - 243x^2 + 27x^4 - x^6)dx = \frac{2k}{3}\left(729x - 81x^3 + \frac{27}{5}x^5 - \frac{1}{7}x^7\right)\Big]_0^3$$

$$= 2k\left(729 - 729 + \frac{2187}{5} - \frac{729}{7}\right) = \frac{23328k}{35}$$

$$M_x = \int_{-3}^{3}\int_{0}^{9-x^2} ky^2 y dy\, dx = \int_{-3}^{3} \frac{k}{4}y^4\Big]_0^{9-x^2} dx = \frac{k}{4}\int_{-3}^{3}(9-x^2)^4 dx$$

$$= \frac{k}{4}\int_{-3}^{3}(6561 - 2916x^2 + 486x^4 - 36x^6 + x^8)dx$$

$$= \frac{k}{4}\left(6561x - 972x^3 + \frac{486}{5}x^5 - \frac{36}{7}x^7 + \frac{1}{9}x^9\right)\Big]_{-3}^3$$

$$= \frac{k}{4}\cdot 2\left(19683 - 26244 + \frac{118098}{5} - \frac{78732}{7} + 2187\right) = \frac{139968k}{35}$$

$$M_y = \int_{-3}^{3}\int_{0}^{9-x^2} ky^2 x dy\, dx = \int_{-3}^{3} \frac{k}{3}(9-x^2)^3 x dx = 0 \quad \text{(odd function)}$$

Then

$$\bar{x} = \frac{M_y}{m} = 0, \qquad \bar{y} = \frac{M_x}{m} = \frac{139968k}{35} \div \frac{23328k}{35} = 6.$$

Therefore, the center of mass is: $(\bar{x}, \bar{y}) = (0, 6)$.

Remark: This is the most complicated problem in terms of integration and algebra. Those who got the setup integrals right would get 12 points.

6. (15 points) Set up double or triple integrals but do *not* evaluate them:

(a) the surface area of the graph of f over the region R:

$$f(x,y) = x^3 - 3xy + y^3, \qquad R = \{(x,y): 0 \le x \le 4, 0 \le y \le x\}.$$

Solution:

Notice that $f_x' = 3x^2 - 3y = 3(x^2 - y)$, $f_y' = -3x + 3y^2 = 3(y^2 - x)$.
Therefore, the surface area is:

$$S = \iint_R \sqrt{1 + (f_x')^2 + (f_y')^2}\, dA = \int_0^4 \int_0^x \sqrt{1 + 9(x^2 - y)^2 + 9(y^2 - x)^2}\, dy\, dx.$$

(b) the volume of the ellipsoid given by $4x^2 + 4y^2 + z^2 = 16$.

Solution: This is Example 2 on page 1029.

Notice that the solid region is an ellipsoid with center $(0,0,0)$.
Let $z = 0$. Then the ellipsoid has trace in the xy-plane ($z = 0$): $4x^2 + 4y^2 = 16$, which is a circle. So, $-2 \le x \le 2, -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}$.
Now z is bounded by two surfaces:
$-\sqrt{16 - 4x^2 - 4y^2} \le z \le \sqrt{16 - 4x^2 - 4y^2}$.
Therefore, the volume is:

$$V = \iiint_Q dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{16-4x^2-4y^2}}^{\sqrt{16-4x^2-4y^2}} dz\, dy\, dx$$

Remark: One may use symmetry to evaluate the volume by using the first octant.

(c) the mass of the solid of given density bounded by the graphs of the equations:

$$\rho(x,y,z) = kz, \qquad Q: x \ge 0, y \ge 0, z \ge 0, y = x, z = 9 - x^2.$$

Solution:

Notice that the trace of $z = 9 - x^2$ of Q in the xy-plane ($z = 0$) is $z = 0, x = 3$.
Then region Q can be represented as: $0 \le x \le 3, 0 \le y \le x$ and $0 \le z \le 9 - x^2$.
Therefore, the mass of the solid is:

$$m = \iiint_Q \rho(x,y,z)\, dV = \int_0^3 \int_0^x \int_0^{9-x^2} kz\, dz\, dy\, dx$$

7. (10 points) Determine whether each statement is *true* or *false*.

(1) The following two integrals are equal:
$\int_a^b \int_c^d f(x)g(y)dy\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_c^d f(y)dy\right)$. Answer:__ *false*___

(2) The following two iterated integrals are equal:
$\int_0^2 \int_0^1 f(x,y)dx\,dy = \int_0^1 \int_0^2 f(x,y)dy\,dx$. Answer:___ *true*___

(3) If $f \leq g$ for all (x,y) in R, and both f and g are continuous over R, then $\iint_R f(x,y)dA \leq \iint_R g(x,y)dA$. Answer:___ *true*___

(4) The volume of the sphere $x^2+y^2+z^2=1$ is given by the integral $V = 8\int_0^1 \int_0^1 \sqrt{1-x^2-y^2}dy\,dx$. Answer:__ *false*___

(5) If $\iint_R f(r,\theta)dA > 0$, then $f(r,\theta) > 0$ for all (r,θ) in R. Answer:__ *false*___