Test #3 Solution

Date: 04/02/2013 Name: ____________________

NOTE: You must show all work to earn credit.

1. (10 points) Find the limit of the function if it exists:

$$\lim_{(x,y)\to(0,0)} \frac{4x^2y}{x^4+y^2}$$

Solution:

We need to examine at least two paths through which (x,y) approaches $(0,0)$, and see what the function approaches.

Now, let's consider: $y = x^2$, and $y = 2x^2$. Notice that when x approaches 0, y also approaches 0. (We cannot take path $y = x^2 + 1$, since y will approaches 1 when x approaches 0.)

(1) $y = x^2$. Then

$$\lim_{(x,y)\to(0,0)} \frac{4x^2y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{4x^2(x^2)}{x^4+(x^2)^2} = \lim_{(x,y)\to(0,0)} \frac{4x^4}{x^4+x^4} = 2.$$

(2) $y = 2x^2$. Then

$$\lim_{(x,y)\to(0,0)} \frac{4x^2y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{4x^2(2x^2)}{x^4+(2x^2)^2} = \lim_{(x,y)\to(0,0)} \frac{8x^4}{x^4+4x^4} = \frac{8}{5}.$$

Therefore, the limit does *not* exist, as $2 \neq 8/5$.

2. (10 points) Show that the function $z = e^x \sin y$ satisfies Laplace's equation:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

Proof:

$$\frac{\partial z}{\partial x} = e^x \sin y, \qquad \frac{\partial z}{\partial y} = e^x \cos y.$$

$$\frac{\partial^2 z}{\partial x^2} = e^x \sin y, \qquad \frac{\partial^2 z}{\partial y^2} = -e^x \sin y.$$

Therefore, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^x \sin y - e^x \sin y = 0$.

Q.E.D. (quod erat demonstrandum)

3. (10 points) Find the directional derivative of the function at P in the direction of $\vec{v}$.

$$f(x,y) = x^2 y, \qquad P(2,1), \qquad \vec{v} = 3\vec{\imath} - 4\vec{\jmath}.$$

Solution:

First, we need to normalize the vector in order to find the directional derivative:

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{3\vec{\imath} - 4\vec{\jmath}}{\|3\vec{\imath} - 4\vec{\jmath}\|} = \frac{3\vec{\imath} - 4\vec{\jmath}}{\sqrt{3^2 + (-4)^2}} = \frac{3\vec{\imath} - 4\vec{\jmath}}{5} = \frac{3}{5}\vec{\imath} - \frac{4}{5}\vec{\jmath}.$$

Second, we need to find the gradient of the function:

$$\nabla f(x,y) = 2xy\vec{\imath} + x^2\vec{\jmath}, \qquad so\ \nabla f(2,1) = 4\vec{\imath} + 4\vec{\jmath}.$$

Therefore, the directional derivative can be calculated by

$$D_{\vec{u}} f(2,1) = \nabla f(2,1) \cdot \vec{u} = (4\vec{\imath} + 4\vec{\jmath}) \cdot \left(\frac{3}{5}\vec{\imath} - \frac{4}{5}\vec{\jmath}\right) = \frac{12}{5} - \frac{16}{5} = -\frac{4}{5}.$$

4. (15 points) Find an equation of the tangent plane and parametric equations of the normal line to the surface at the given point.

$$z = x^2 - 2xy + y^2, \qquad (1,2,1)$$

Solution:

We need to find the gradient of the function, which gives the normal vector of the tangent plane and directional vector of the normal line. In order to do that, let us write the function in the form: $F(x,y,z) = x^2 - 2xy + y^2 - z = 0$. Then $\nabla F(x,y,z) = (2x - 2y)\vec{\imath} + (-2x + 2y)\vec{\jmath} - \vec{k}$. So, $\nabla F(1,2,1) = -2\vec{\imath} + 2\vec{\jmath} - \vec{k}$

If we take $\vec{n} = -2\vec{\imath} + 2\vec{\jmath} - \vec{k}$ as the normal vector of the tangent plane, then we get an equation of the plane:

$$-2(x-1) + 2(y-2) - (z-1) = 0, \text{ or } 2x - 2y + z + 1 = 0.$$

If we take $\vec{v} = -2\vec{\imath} + 2\vec{\jmath} - \vec{k}$ as the directional vector of the normal line, then we get a system of parametric equations of the line:

$$x = -2t + 1,$$
$$y = 2t + 2,$$
$$z = -t + 1.$$

5. (15 points) Examine the function for relative extrema and saddle point.

$$f(x,y) = 8x^3 - 6xy + y^3$$

Solution:

First, we need to find the critical points of the function:

$$f_x'(x,y) = 24x^2 - 6y = 0, \qquad f_y'(x,y) = -6x + 3y^2 = 0.$$

So, $y = 4x^2, x = \frac{1}{2}y^2$. Then substitute y into the second equation to solve for x: $x = \frac{1}{2}(4x^2)^2 = 8x^4$, or $x(8x^3 - 1) = 0$. Therefore we have two solutions: $x = 0$, or $x = \frac{1}{2}$. Then $y = 0$, or $y = 1$, respectively. We have found two critical points: $(0,0)$ and $\left(\frac{1}{2}, 1\right)$.

Next, we use second partials test to examine those two critical points.

$$f_{xx}''(x,y) = 48x, \qquad f_{xy}''(x,y) = -6, \qquad f_{yy}''(x,y) = 6y.$$

(1) At $(0,0)$, $d = f_{xx}''(0,0)f_{yy}''(0,0) - \left[f_{xy}''(0,0)\right]^2 = 0 - (-6)^2 = -36 < 0$. So, it is a saddle point.

(2) At $\left(\frac{1}{2},1\right)$, $d = f_{xx}''\left(\frac{1}{2},1\right) f_{yy}''\left(\frac{1}{2},1\right) - \left[f_{xy}''\left(\frac{1}{2},1\right)\right]^2 = 24 \times 6 - (-6)^2 = 108$. So, $d > 0$ and $f_{xx}''\left(\frac{1}{2},1\right) = 24 > 0$. Hence it is a minimum. $f\left(\frac{1}{2},1\right) = -2$.

6. (10 points) A manufacturer has an order for 1000 units of chairs that can be produced at two locations. Let x_1 and x_2 be the numbers of units produced at the two locations. The cost function is

$$C = 0.25{x_1}^2 + 10x_1 + 0.15{x_2}^2 + 12x_2.$$

Find the number that should be produced at each location to meet the order and minimize the cost.

Solution:

We need to minimize the cost function subject to the constraint $x_1 + x_2 = 1000$:

$$\frac{\partial C}{\partial x_1} = 0.5x_1 + 10 = \lambda, \qquad \frac{\partial C}{\partial x_2} = 0.3x_2 + 12 = \lambda.$$

Therefore, we get a set of equations:

$$0.5x_1 + 10 = \lambda,$$
$$0.3x_2 + 12 = \lambda,$$
$$x_1 + x_2 = 1000.$$

Solving for x_1 and x_2: $x_1 = 377.5$, and $x_2 = 622.5$.

So, $C(377.5, 622.5) = 0.25(377.5)^2 + 10(377.5) + 0.15(622.5)^2 + 12(622.5) =$ 104,997.50 is the minimum cost.

NOTE: $x_1 = 378$ and $x_2 = 622$ are acceptable, so are $x_1 = 377$ and $x_2 = 623$.

7. (10 points) Use Lagrange multipliers to minimize function $f(x, y) = 3x + y + 10$ subject to the constraint $x^2 y = 12$, where $x > 0$.

Solution:

According to the method of Lagrange multipliers:

$$\nabla f(x, y) = \lambda \nabla (x^2 y).$$

So, we have the following equations:

$$3\vec{\imath} + \vec{\jmath} = \lambda(2xy\vec{\imath} + x^2\vec{\jmath}), \qquad x^2 y = 12, \qquad x > 0.$$

In other words,

$$3 = 2\lambda xy, \qquad 1 = \lambda x^2, \qquad x^2 y = 12, \qquad x > 0.$$

$$3x = 2y, \qquad x^2 y = 12, \qquad x > 0.$$

$$3x^3 = 24, \qquad \text{so } x = 2, y = 3.$$

Therefore, $f(2, 3) = 3(2) + 3 + 10 = 19$ is the minimum.

NOTE: Let's try a few (x, y) that satisfies the constraint and see if 19 is the minimum.
$f(1, 12) = 3 + 12 + 10 = 25, f(3, 4/3) = 9 + 4/3 + 10 = 20.33$.

8. (20 points) Determine whether each statement is *true* or *false*.

(1) The limit $\lim_{(x,y)\to(0,0)} \frac{6x^2 y}{x^2 + y^2}$ does not exist. Answer:__ *false* ___

(2) A vertical line can intersect the graph of $z = f(x, y)$ at most once. Answer:___ *true* ___

(3) If f and g are continuous functions of x and y, then $f - g$ is also a continuous function. Answer:___ *true* ___

(4) If $f(x, y)$ always has continuous partial derivatives f''_{xy} and f''_{yx}, then $f''_{xy}(x, y) = f''_{yx}(x, y)$. Answer:___ *true* ___

(5) If $z = f(x)g(y)$, then $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = f'(x) + g'(y)$. Answer:__ *false* ___

(6) If $f(x, y) = x + y$, then $-1.5 \le D_{\vec{u}} f(x, y) \le 1.5$. Answer:___ *true* ___

(7) If $D_{\vec{u}} f(x_0, y_0) = c$ for any unit vector $\vec{u}$, then $c = 0$. Answer:___ *true* ___

(8) If f has a relative maximum at (x_0, y_0), then $f'_x(x_0, y_0) = 0$ and $f'_y(x_0, y_0) = 0$. Answer:__ *false* ___

(9) The method of Lagrange multipliers is used to find critical points. Answer:___ *true* ___

(10) The second partials test is used to find critical points. Answer:__ *false* ___