Test #2 Solution

Date: 02/19/2013 Name: ______________________

NOTE: You must show all work to earn credit.

1. (12 points) Find the limit:

$$\lim_{t\to 0}\left(\frac{3t}{\sin 2t}\,\vec{\imath}+\frac{1-\cos t}{t^2}\,\vec{\jmath}+(1-t)^{1/t}\,\vec{k}\right)$$

Solution:

$$\lim_{t\to 0}\left(\frac{3t}{\sin 2t}\,\vec{\imath}+\frac{1-\cos t}{t^2}\,\vec{\jmath}+(1-t)^{\frac{1}{t}}\,\vec{k}\right)$$
$$=\lim_{t\to 0}\left(\frac{3t}{\sin 2t}\right)\vec{\imath}+\lim_{t\to 0}\left(\frac{1-\cos t}{t^2}\right)\vec{\jmath}+\lim_{t\to 0}\left((1-t)^{\frac{1}{t}}\right)\vec{k}$$
$$=\lim_{t\to 0}\left(\frac{3}{2\cos 2t}\right)\vec{\imath}+\lim_{t\to 0}\left(\frac{\sin t}{2t}\right)\vec{\jmath}+\lim_{t\to 0}\left((1-t)^{-\frac{1}{t}}\right)^{-1}\vec{k}$$
$$=\frac{3}{2}\vec{\imath}+\frac{1}{2}\vec{\jmath}+e^{-1}\vec{k}$$

2. (12 points) Find the indefinite integral:

$$\int\left(\ln t\,\vec{\imath}+\frac{1}{1+t^2}\vec{\jmath}+\frac{t}{1+t^2}\vec{k}\right)dt$$

Solution:

$$\int\left(\ln t\,\vec{\imath}+\frac{1}{1+t^2}\vec{\jmath}+\frac{t}{1+t^2}\vec{k}\right)dt$$
$$=\left(\int \ln t\,dt\right)\vec{\imath}+\left(\int\frac{1}{1+t^2}dt\right)\vec{\jmath}+\left(\int\frac{t}{1+t^2}dt\right)\vec{k}$$
$$=\left[t\ln t-\int t\cdot d(\ln t)\right]\vec{\imath}+(\arctan t)\vec{\jmath}+\left[\frac{1}{2}\ln(1+t^2)\right]\vec{k}$$
$$=\left[t\ln t-\int dt\right]\vec{\imath}+(\arctan t)\vec{\jmath}+\left[\frac{1}{2}\ln(1+t^2)\right]\vec{k}$$
$$=(t\ln t-t)\,\vec{\imath}+(\arctan t)\,\vec{\jmath}+\left[\frac{1}{2}\ln(1+t^2)\right]\vec{k}+\vec{C}$$

3. (12 points) Find $\vec{r}(t)$ for the given conditions:

$$\vec{r}'(t) = 2t\vec{\imath} + e^t\vec{\jmath} + e^{-t}\vec{k}, \qquad \vec{r}(0) = \vec{\imath} + 3\vec{\jmath} - 5\vec{k}.$$

Solution:

$$\vec{r}(t) = \int \left(2t\vec{\imath} + e^t\vec{\jmath} + e^{-t}\vec{k}\right) dt = t^2\vec{\imath} + e^t\vec{\jmath} - e^{-t}\vec{k} + \vec{C}.$$

Notice that $\vec{r}(0) = \vec{\imath} + 3\vec{\jmath} - 5\vec{k}$, and $\vec{r}(0) = 0^2\vec{\imath} + e^0\vec{\jmath} - e^{-0}\vec{k} + \vec{C} = \vec{\jmath} - \vec{k} + \vec{C}$.

So, $\vec{\jmath} - \vec{k} + \vec{C} = \vec{\imath} + 3\vec{\jmath} - 5\vec{k}$. Solve for $\vec{C}$:

$$\vec{C} = \vec{\imath} + 3\vec{\jmath} - 5\vec{k} - \vec{\jmath} + \vec{k} = \vec{\imath} + 2\vec{\jmath} - 4\vec{k}.$$

It follows that

$$\vec{r}(t) = t^2\vec{\imath} + e^t\vec{\jmath} - e^{-t}\vec{k} + \vec{C} = t^2\vec{\imath} + e^t\vec{\jmath} - e^{-t}\vec{k} + \vec{\imath} + 2\vec{\jmath} - 4\vec{k}.$$

Therefore, $\vec{r}(t) = (t^2 + 1)\vec{\imath} + (e^t + 2)\vec{\jmath} - (e^{-t} + 4)\vec{k}$.

4. (20 points) Let $\vec{r}(t) = \sin t\,\vec{\imath} + \cos t\,\vec{\jmath} + t\vec{k}$ and $\vec{u}(t) = \sin t\,\vec{\imath} + \cos t\,\vec{\jmath} + \frac{1}{t}\vec{k}$. Find

(1) $\vec{r}''(t)$
(2) $D_t[\vec{r}(t) \cdot \vec{u}(t)]$
(3) $D_t[\vec{r}(t) \times \vec{u}(t)]$
(4) $D_t[\|\vec{r}(t)\|], t > 0$

Solution:

(1) $\vec{r}'(t) = \cos t\,\vec{\imath} - \sin t\,\vec{\jmath} + \vec{k}$, so
$\vec{r}''(t) = -\sin t\,\vec{\imath} - \cos t\,\vec{\jmath}$

(2) $\vec{r}(t) \cdot \vec{u}(t) = \left(\sin t\,\vec{\imath} + \cos t\,\vec{\jmath} + t\vec{k}\right) \cdot \left(\sin t\,\vec{\imath} + \cos t\,\vec{\jmath} + \frac{1}{t}\vec{k}\right)$
$= \sin^2 t + \cos^2 t + 1 = 2$. Therefore, $D_t[\vec{r}(t) \cdot \vec{u}(t)] = D_t[2] = 0$

(3) $\vec{r}(t) \times \vec{u}(t) = \left(\sin t\,\vec{\imath} + \cos t\,\vec{\jmath} + t\vec{k}\right) \times \left(\sin t\,\vec{\imath} + \cos t\,\vec{\jmath} + \frac{1}{t}\vec{k}\right)$

$$= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \sin t & \cos t & t \\ \sin t & \cos t & 1/t \end{vmatrix} = \left(\frac{1}{t} - t\right)\cos t\,\vec{\imath} + \left(t - \frac{1}{t}\right)\sin t\,\vec{\jmath}$$

Therefore, $D_t[\vec{r}(t) \times \vec{u}(t)] = D_t\left[\left(\frac{1}{t} - t\right)\cos t\,\vec{\imath} + \left(t - \frac{1}{t}\right)\sin t\,\vec{\jmath}\right] =$
$[(-t^{-2} - 1)\cos t - (t^{-1} - t)\sin t]\,\vec{\imath} + [(1 + t^{-2})\sin t + (t - t^{-1})\cos t]\,\vec{\jmath}$.

(4) $\|\vec{r}(t)\| = \sqrt{(\sin t)^2 + (\cos t)^2 + t^2} = \sqrt{1 + t^2}$. So,

$$D_t[\|\vec{r}(t)\|] = \left(\sqrt{1 + t^2}\right)' = \frac{1}{2}(1 + t^2)^{-1/2}(1 + t^2)' = \frac{t}{\sqrt{1 + t^2}}.$$

5. (12 points) A baseball field is surrounded by a 10-foot-high fence, and it is 350 feet away from home plate to the fence. A baseball is hit 3 feet above the ground at 100 feet per second and at an angle of 45° with respect to the ground. Will it be a home run? Justify your answer by using the position function for a projectile:

$$\vec{r}(t) = (v_0 \cos\theta)t\,\vec{\imath} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\vec{\jmath}$$

Solution:

It follows from the given conditions that we have:
$v_0 = 100$ feet/sec, $\theta = 45°$, $h = 3$ feet, $g = 32$ feet/sec^2.

$\vec{r}(t) = (100\cos 45°)t\,\vec{\imath} + [3 + (100\sin 45°)t - 16t^2]\,\vec{\jmath}$
$= 50\sqrt{2}t\,\vec{\imath} + [3 + 50\sqrt{2}t - 16t^2]\,\vec{\jmath}$
Now we know that $x(t) = 50\sqrt{2}t, y(t) = 3 + 50\sqrt{2}t - 16t^2$.
Since $x = 350$ feet, $50\sqrt{2}t = 350$. Solve for t:

$$t = 350/(50\sqrt{2}) = 7\sqrt{2}/2 \approx 4.95.$$

Substitute $t = \frac{7\sqrt{2}}{2}$ into $y(t) = 3 + 50\sqrt{2}t - 16t^2$ to calculate the height of the ball and see if it clears the 10-foot-high fence for a home run:
$y(7\sqrt{2}/2) = 3 + (50\sqrt{2})(7\sqrt{2}/2) - 16(7\sqrt{2}/2)^2$
$= 3 + 350 - 392 = -39$ feet at 4.95 seconds.

This implies that the ball does not clear the fence and it is not a home run. In fact, it hits the ground prior to 4.95 seconds.

6. (12 points) Find the curvature and radius of curvature of the plane curve $y = \sqrt{4 - 9x^2}$ at the given value of $x = 0$.

Solution:

We can use the curvature formula:

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}}.$$

Notice that

$$y' = \frac{-9x}{\sqrt{4 - 9x^2}}, y'' = \frac{-9\sqrt{4 - 9x^2} - 81x^2/\sqrt{4 - 9x^2}}{(4 - 9x^2)} = \frac{-36}{(4 - 9x^2)^{3/2}}.$$

When $x = 0$, we have $y' = 0, y'' = -9/2$. Therefore,

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{\frac{9}{2}}{[1 + (0)^2]^{3/2}} = \frac{9}{2}.$$

The radius of curvature is:

$$r = \frac{1}{K} = \frac{2}{9}.$$

7. (20 points) Determine whether each statement is *true* or *false*.

(1) The vector-valued function $\vec{r}(t) = t^2\vec{i} + t\sin t\,\vec{j} + t\cos t\,\vec{k}$ lies on the paraboloid $x + y^2 + z^2 = 0$. Answer:__ *false*___

(2) If $\vec{r}$ is a vector-valued function, then $\vec{r}$ and $\vec{r}'$ are orthogonal to each other. Answer:__ *false*___

(3) If $\vec{r}$ and $\vec{u}$ are differentiable vector-valued functions of t, then $D_t[\vec{r}(t)\cdot\vec{u}(t)] = \vec{r}'(t)\cdot\vec{u}'(t)$. Answer:__ *false*___

(4) The acceleration of an object is the derivative of the speed. Answer:__ *false*___

(5) If $a_{\vec{N}} = 0$ for a moving object, then the object is moving in a straight line. Answer:___*true*___

(6) The curvature of a line is 0. Answer:___*true*___

(7) The curvature of a circle may be the radius of the circle. Answer:___*true*___

(8) The arc length of a space curve does not depend on the parameter. Answer:___*true*___

(9) The derivative of the unit vector is a unit vector. Answer:__ *false*___

(10) $\int_{-a}^{a}\left[(\sin^{2013} t)\vec{i} + (t\sin^2 t)\vec{j} + (t^{19})\vec{k}\right]dt = 0.$ Answer:__ *false*___