Test #1 Solution

Date: 02/05/2013 Name: ____________________

NOTE: You must show all work to earn credit.

1. (20 points) Let $\vec{u} = \overrightarrow{PQ}$ and $\vec{v} = \overrightarrow{PR}$, where $P = (-2, -1, 1)$, $Q = (2, -4, 1)$, and $R = (1, -1, 5)$. Find (1) the norm of $\vec{u} - \vec{v}$, (2) the angle between $\vec{u}$ and $\vec{v}$, (3) the dot product of $\vec{u}$ and $\vec{v}$, and (4) the cross product of $\vec{u}$ and $\vec{v}$.

Solution:

It's easy to get $\vec{u} = \overrightarrow{PQ} = \langle 4, -3, 0 \rangle$, and $\vec{v} = \overrightarrow{PR} = \langle 3, 0, 4 \rangle$.

(1) $\vec{u} - \vec{v} = \langle 4, -3, 0 \rangle - \langle 3, 0, 4 \rangle = \langle 1, -3, -4 \rangle$. Therefore, the norm is equal to

$$\|\vec{u} - \vec{v}\| = \sqrt{1^2 + (-3)^2 + (-4)^2} = \sqrt{26}.$$

(2) Let θ be the angle between $\vec{u}$ and $\vec{v}$. Then

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{\langle 4, -3, 0 \rangle \cdot \langle 3, 0, 4 \rangle}{\sqrt{4^2 + (-3)^2 + 0^2}\sqrt{3^2 + 0^2 + 4^2}} = \frac{12}{25}.$$

Therefore, $\theta = \arccos\frac{12}{25} \approx 61.31°$.

(3) $\vec{u} \cdot \vec{v} = \langle 4, -3, 0 \rangle \cdot \langle 3, 0, 4 \rangle = 12$.

(4) $$\vec{u} \times \vec{v} = \langle 4, -3, 0 \rangle \times \langle 3, 0, 4 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -3 & 0 \\ 3 & 0 & 4 \end{vmatrix} = -12\vec{i} - 16\vec{j} + 9\vec{k}.$$

2. (12 points) Let $\vec{u} = \langle 3, -2, 1\rangle$ and $\vec{v} = \langle 3, 4, 12\rangle$. Find (1) the projection of $\vec{u}$ onto $\vec{v}$, and (2) the vector component of $\vec{u}$ orthogonal to $\vec{v}$.

Solution:

(1) The projection of $\vec{u} = \langle 3, -2, 1\rangle$ onto $\vec{v} = \langle 3, 4, 12\rangle$ is

$$\text{proj}_{\vec{v}}\, \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\right)\vec{v} = \left(\frac{\langle 3, -2, 1\rangle \cdot \langle 3, 4, 12\rangle}{\|\langle 3, 4, 12\rangle\|^2}\right)\langle 3, 4, 12\rangle = \frac{13}{169}\langle 3, 4, 12\rangle.$$

Therefore, $\text{proj}_{\vec{v}}\, \vec{u} = \langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\rangle$.

(2) The vector component of $\vec{u}$ orthogonal to $\vec{v}$ is

$$\vec{u} - \text{proj}_{\vec{v}}\, \vec{u} = \langle 3, -2, 1\rangle - \langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\rangle = \langle \frac{36}{13}, -\frac{30}{13}, \frac{1}{13}\rangle$$

3. (12 points) Find a set of symmetric equations for the line that passes through the point $(0, 1, 4)$, and is perpendicular to $\vec{u} = \langle 2, -5, 1\rangle$ and $\vec{v} = \langle -3, 1, 4\rangle$.

Solution:

The direction vector of the line is orthogonal to $\vec{u}$ and $\vec{v}$. So we can take $\vec{u} \times \vec{v}$ as a direction vector.

$$\vec{u} \times \vec{v} = \langle 2, -5, 1\rangle \times \langle -3, 1, 4\rangle = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & -5 & 1 \\ -3 & 1 & 4 \end{vmatrix} = -21\vec{\imath} - 11\vec{\jmath} - 13\vec{k}.$$

Therefore, a set of symmetric equations for the line can be as follows:

$$\frac{x - 0}{-21} = \frac{y - 1}{-11} = \frac{z - 4}{-13}.$$

Note: One can get other sets of equations, e.g., $\frac{x}{21} = \frac{y-1}{11} = \frac{z-4}{13}$, or $\frac{x}{42} = \frac{y-1}{22} = \frac{z-4}{26}$.

4. (12 points) Find an equation of the plane that passes through the point $(-2, 3, 1)$ and is perpendicular to $\langle 3, -1, 1 \rangle$.

Solution:

The normal vector of the plane is $\langle 3, -1, 1 \rangle$, and the point in the plane is $(-2, 3, 1)$. Therefore, an equation of this plane in standard form can be:

$$3(x + 2) - (y - 3) + (z - 1) = 0.$$

Note: One can simplify this equation to get an equation in general form:

$$3x - y + z + 8 = 0.$$

5. (12 points) An object is pulled 8 feet across the floor using a force of 75 pounds. The direction of the force is $30°$ above the horizontal. Find the work done.

Solution:

Let $\vec{F}$ denote the force, $\overrightarrow{PQ}$ be the vector of the displacement (directed distance), and θ be the angle between $\vec{F}$ and $\overrightarrow{PQ}$.

The work can be evaluated by the formula:

$$W = \left\|\text{proj}_{\overrightarrow{PQ}} \vec{F}\right\| \left\|\overrightarrow{PQ}\right\| = \left(\left\|\vec{F}\right\| \cos\theta\right) \left\|\overrightarrow{PQ}\right\| = (75 \cdot \cos 30°) \cdot 8 = 300\sqrt{3}$$

Therefore, the work done is $300\sqrt{3} \approx$ 519.6 foot-pounds.

6. (12 points) Convert the point $(\sqrt{3}, 0, -1)$ in rectangular coordinates into (1) cylindrical coordinates, and (2) spherical coordinates.

Solution:

Given $x = \sqrt{3}, y = 0, z = -1$.

(1) In cylindrical coordinates (r, θ, z), $r = \sqrt{x^2 + y^2} = \sqrt{3}$, $\theta = \arctan\frac{y}{x} = 0$, and $z = -1$. Therefore, $(r, \theta, z) = (\sqrt{3}, 0, -1)$, or $(\sqrt{3}, 0°, -1)$.

(2) In spherical coordinates (ρ, θ, φ), $\rho = \sqrt{x^2 + y^2 + z^2} = 2$, $\theta = \arctan\frac{y}{x} = 0$, and $\varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \arccos\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$. Therefore,

$$(\rho, \theta, \varphi) = \left(2, 0, \frac{2\pi}{3}\right), \text{ or } (2, 0°, 120°).$$

7. (20 points) Determine whether each statement is *true* or *false*.

(a) If $\vec{u}$ and $\vec{v}$ are orthogonal to $\vec{w}$, then $\vec{u} + \vec{v}$ is orthogonal to $\vec{w}$. Answer:___*true*___

(b) If the dot product $\vec{u} \cdot \vec{v} = 0$, then $\vec{u}$ is orthogonal to $\vec{v}$. Answer:___*true*___

(c) If $\vec{u}$ and $\vec{v}$ have the same magnitude but opposite directions, then $\vec{u} + \vec{v} = 0$. Answer:___*false*__

(d) Every pair of lines in space are either intersecting or parallel. Answer:___*false*__

(e) In spherical coordinates, the equation $\theta = c$ represents an entire plane. Answer:___*false*__

(f) In cylindrical coordinates, the equation $\theta = c$ represents a line. Answer:___*false*__

(g) The rectangular equation $x^2 + y^2 + z^2 - 9z = 0$ can be converted to spherical equation $\rho = 9 \sin\varphi$. Answer:___*false*__

(h) $x^2 + 2y^2 - 3z^2 = 1$ represents an equation of elliptic cone. Answer:___*false*__

(i) The distance between $(2, 3, 1)$ and $x + y + z - 4 = 0$ is greater than 1. Answer:___*true*___

(j) Given two nonzero orthogonal vectors, the dot product of those two vectors is $\vec{0}$. Answer:___*false*__