Quiz #9 Solution

Date: 04/30/2013 Name: ____________________

NOTE: You must show all work to earn credit.

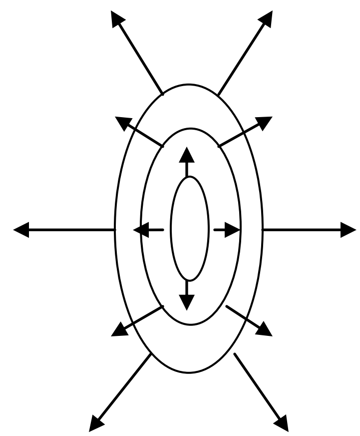
1. (10 points) Compute $\|\vec{F}\|$ and sketch several representative vectors in the vector field $\vec{F}(x,y) = 4x\,\vec{\imath} + y\,\vec{\jmath}$.

Solution:

We will draw the vectors with the same norm $\|\vec{F}\| = \sqrt{16x^2 + y^2} = c$, where c is a constant. This equation is equivalent to:

$$\frac{x^2}{c^2/16} + \frac{y^2}{c^2} = 1$$

which is an ellipse.



Next, take $c = 1, 2, 4$. Then we get 3 ellipses, and draw some vectors with norm $1, 2, 4$ on those ellipses, respectively.

2. (8 points) Determine whether the vector field $\vec{F}(x,y) = xe^{x^2y}(2y\vec{\imath} + x\vec{\jmath})$ is conservative. Justify your answer.

Solution:

Notice that $M(x,y) = 2xye^{x^2y}, N(x,y) = x^2e^{x^2y}$.

$$\frac{\partial M}{\partial y} = 2xe^{x^2y} + 2x^3ye^{x^2y}, \qquad \frac{\partial N}{\partial x} = 2xe^{x^2y} + 2x^3ye^{x^2y}.$$

Therefore, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. It follows that $\vec{F}(x,y)$ is conservative.

Next we need to find the potential function $f(x,y)$.

$$f_x'(x,y) = M(x,y) = 2xye^{x^2y}, \text{so } f(x,y) = e^{x^2y} + g(y).$$

Then

$$f_y'(x,y) = x^2e^{x^2y} + g'(y) = N(x,y) = x^2e^{x^2y}, \text{so } g'(y) = 0.$$

We have $g(y) = K$, where K is any constant. Consequently, $f(x,y) = e^{x^2y} + K$.

3. (6 points) Find $\overrightarrow{\text{curl}}\,\vec{F}$ for the vector field at the given point:

$$\vec{F}(x,y,z) = e^x \sin y\,\vec{\imath} - e^x \cos y\,\vec{\jmath}, \qquad (0,0,1).$$

Solution:

$$\overrightarrow{\text{curl}}\,\vec{F}(x,y,z) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x \sin y & -e^x \cos y & 0 \end{vmatrix} = -2e^x \cos y\,\vec{k}.$$

Therefore, $\overrightarrow{\text{curl}}\,\vec{F}(0,0,1) = -2\vec{k}$.

4. (6 points) Find the divergence of the vector field at the given point:

$$\vec{F}(x,y,z) = x^2 z\,\vec{\imath} - 2xz\,\vec{\jmath} + yz\,\vec{k}, \quad (2,-1,3).$$

Solution:

The divergence of the vector field is:

$$\text{div}\,\vec{F}(x,y,z) = 2xz - 0 + y = 2xz + y.$$

Therefore,

$$\text{div}\,\vec{F}(2,-1,3) = 2(2)(3) - 1 = 11.$$