Quiz #8 Solution

Date: 04/16/2013 Name: ____________________

NOTE: You must show all work to earn credit.

1. (8 points) Evaluate the iterated integral by converting to polar coordinates:

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} xy\, dydx.$$

Solution:

Notice that the region R is the semi circle with center $(2, 0)$ and radius 2. This semi circle can be represented by the polar equation: $\rho = 2\cos\theta$. So the region is: $0 \le \theta \le \frac{\pi}{2}$, $0 \le \rho \le 2\cos\theta$. Then

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} xy\, dydx = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} (r\cos\theta)(r\sin\theta)\, rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4} r^4 \cos\theta \sin\theta \Big]_0^{2\cos\theta} d\theta = \int_0^{\frac{\pi}{2}} 4(\cos\theta)^5 \sin\theta\, d\theta$$

$$= -\frac{4\cos^6\theta}{6}\Big]_0^{\frac{\pi}{2}} = \frac{2}{3}.$$

Therefore,

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} xy\, dydx = \frac{2}{3}.$$

2. (6 points) Use a double integral in polar coordinates to find the volume of the solid bounded by the graphs of the equations:

$$z = \sqrt{x^2 + y^2}, \qquad z = 0, \qquad x^2 + y^2 = 25.$$

Solution:

Notice that the region R is the circle with center $(0, 0)$ and radius 5. It can be represented by the polar equation: $\rho = 5$. The region is: $0 \le \theta \le 2\pi$, $0 \le \rho \le 5$. Also, $z = \sqrt{x^2 + y^2}$ can be written in polar form: $z = r$. So the volume is:

$$V = \int_0^{2\pi} \int_0^5 z\, rdrd\theta = \int_0^{2\pi} \int_0^5 r^2 drd\theta = \int_0^{2\pi} \frac{125}{3} d\theta = \frac{250\pi}{3}.$$

3. (8 points) Find I_x, I_y, I_0, $\bar{\bar{x}}$, and $\bar{\bar{y}}$ for the lamina bounded by the graphs of the equations:

$$y = 4 - x^2, \qquad y = 0, \qquad x > 0, \qquad \rho = kx.$$

Solution: Given $\rho = kx$. The region R is: $0 < x \le 2,\ 0 \le y \le 4 - x^2$.

$$m = \int_0^2 \int_0^{4-x^2} kx\, dydx = \int_0^2 k(4x - x^3)dx = 4k$$

$$I_x = \int_0^2 \int_0^{4-x^2} kxy^2\, dydx = \int_0^2 kx\frac{1}{3}(4-x^2)^3 dx = \frac{k}{6}(64) = \frac{32}{3}k$$

$$I_y = \int_0^2 \int_0^{4-x^2} kxx^2\, dydx = \int_0^2 kx^3(4-x^2)dx = \frac{16}{3}k$$

$$I_0 = I_x + I_y = 16k$$

Therefore,

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{\frac{16}{3}k}{4k}} = \sqrt{\frac{4}{3}}, \qquad \bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{\frac{32}{3}k}{4k}} = \sqrt{\frac{8}{3}}.$$

4. (8 points) Find the mass and center of mass of the lamina bounded by the graphs of the equations for the given density:

$$xy = 4, \qquad y = 0, \qquad x = 1, \qquad x = 4, \qquad \rho = kx^2.$$

Solution: Given $\rho = kx^2$. The region R is: $1 < x \le 4,\ 0 \le y \le 4/x$.

$$m = \int_1^4 \int_0^{\frac{4}{x}} kx^2\, dydx = \int_1^4 kx^2\left(\frac{4}{x}\right)dx = \int_1^4 4kxdx = 2kx^2]_1^4 = 30k$$

$$M_x = \int_1^4 \int_0^{\frac{4}{x}} kx^2 y\, dydx = \int_1^4 \frac{kx^2}{2}\left(\frac{4}{x}\right)^2 dx = \int_1^4 8kdx = 24k$$

$$M_y = \int_1^4 \int_0^{\frac{4}{x}} kx^2 x\, dydx = \int_1^4 kx^3\left(\frac{4}{x}\right)dx = \int_1^4 4kx^2 dx = \frac{4}{3}kx^3\Big]_1^4 = 84k$$

Therefore,

$$\bar{x} = \frac{M_y}{m} = \frac{84k}{30k} = \frac{14}{5}, \qquad \bar{y} = \frac{M_x}{m} = \frac{24k}{30k} = \frac{4}{5}.$$