Quiz #7 Solution

Date: 04/09/2013 Name: ____________________

NOTE: You must show all work to earn credit.

1. (5 points) Evaluate the iterated integral:

$$\int_0^\pi \int_0^{\sin x} (1+\cos x)dydx.$$

Solution:

$$\int_0^\pi \int_0^{\sin x} (1+\cos x)dydx = \int_0^\pi (1+\cos x)y\Big]_0^{\sin x} dx$$

$$= \int_0^\pi (1+\cos x)\sin x\, dx = -\int_0^\pi (1+\cos x)d\cos x$$

$$= \left(-\cos x - \frac{1}{2}\cos^2 x\right)\Big]_0^\pi = \left(1-\frac{1}{2}\right) - \left(-1-\frac{1}{2}\right) = 2.$$

2. (10 points) Switch the order of integration and show that both orders yield the same area of the region given by the iterated integral:

$$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dxdy.$$

Solution:

The region is a semi unit circle above the x-axis. It can be viewed as both vertically and horizontally simple. So

$$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dxdy = \int_{-1}^1 \int_0^{\sqrt{1-x^2}} dydx$$

(1) The first iterated integral

$$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dxdy = \int_0^1 2\sqrt{1-y^2}dy = \int_0^{\frac{\pi}{2}} 2\cos\theta\cos\theta\, d\theta \text{, where } y = \sin\theta$$

$$= \int_0^{\frac{\pi}{2}} (\cos 2\theta + 1)d\theta = \left(\frac{1}{2}\sin 2\theta + \theta\right)\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

(2) The second iterated integral

$$\int_{-1}^1 \int_0^{\sqrt{1-x^2}} dydx = \int_{-1}^1 \sqrt{1-x^2}\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\cos\theta\, d\theta \text{, where } y = \sin\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\cos\theta\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\cos 2\theta + 1}{2}\right) d\theta = \left(\frac{1}{4}\sin 2\theta + \frac{\theta}{2}\right)\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

Therefore, both double integrals will result in the same value: $\frac{\pi}{2}$.

3. (7 points) Do Exercise 27 of section 14.2.

Solution:

Notice that the region is horizontally simple: $0 \le y \le 1$, and $0 \le x \le y$. Therefore, the volume is

$$V = \int_0^1 \int_0^y (1 - xy)\,dx\,dy = \int_0^1 \left(x - \frac{1}{2}x^2 y\right)\Bigg]_0^y dy$$

$$= \int_0^1 \left(y - \frac{1}{2}y^3\right) dy = \left(\frac{1}{2}y^2 - \frac{1}{8}y^4\right)\Bigg]_0^1 = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}.$$

4. (8 points) Evaluate the integral over the region R:

$$\iint_R \frac{y}{1+x^2}\,dA$$

R: region bounded by $y = 0, y = \sqrt{x}, x = 4$.

Solution:

Notice that the region is vertically simple.

$$\iint_R \frac{y}{1+x^2}\,dA = \int_0^4 \int_0^{\sqrt{x}} \frac{y}{1+x^2}\,dy\,dx$$

$$= \frac{1}{2}\int_0^4 \frac{y^2}{1+x^2}\Bigg]_0^{\sqrt{x}} dx = \frac{1}{2}\int_0^4 \frac{x}{1+x^2}\,dx$$

$$= \frac{1}{4}\ln(1+x^2)\Bigg]_0^4 = \frac{1}{4}\ln 17.$$

Therefore, the integral is $\frac{1}{4}\ln 17$.