Quiz #4 Solution

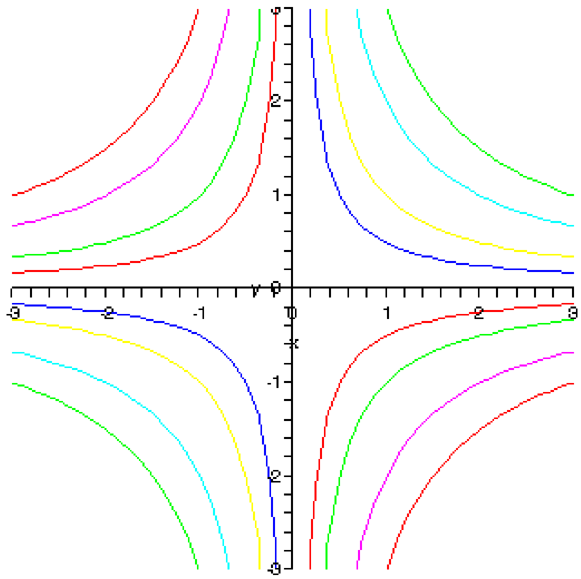
Date: 02/26/2013 Name: ____________________

NOTE: You must show all work to earn credit.

1. (8 points) Describe the level curves of the function $f(x, y) = xy$. Sketch the level curves for the given c-values: $c = \pm1, \pm2, \pm3, \pm4, \pm5, \pm6$.

Solution:

Given $f(x, y) = xy$. The level curves are hyperbolas of the form $xy = c$, where $c = \pm1, \pm2, \pm3, \pm4, \pm5, \pm6$. The pictures of these 12 pairs of curves are hyperbolas that have x- and y-axis as asymptotes.



2. (8 points) Find the limit (if it exists). If the limit does not exist, explain why.

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz + xz}{x^2 + y^2 + z^2}$$

Solution:

Let's consider the following two paths:

(1) $x = y = 0$;

(2) $x = y = z$.

Path (1):

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz + xz}{x^2 + y^2 + z^2} = \lim_{(x,y,z)\to(0,0,0)} \frac{0\cdot 0 + 0\cdot z + 0\cdot z}{0^2 + 0^2 + z^2} = \lim_{(x,y,z)\to(0,0,0)} \frac{0}{z^2} = 0$$

Path (2):

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz + xz}{x^2 + y^2 + z^2} = \lim_{(x,y,z)\to(0,0,0)} \frac{z\cdot z + z\cdot z + z\cdot z}{z^2 + z^2 + z^2} = \lim_{(x,y,z)\to(0,0,0)} \frac{3z^2}{3z^2} = 1$$

Since the limits along two different paths are different, the limit does not exist.

3. (8 points) Consider $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy}$.
   (a) Determine (if possible) the limit along any line of the form $y = ax$.
   (b) Determine (if possible) the limit along the parabola $y = x^2$.
   (c) Does the limit exist? Explain.

Solution:

(a) For $y = ax$, if $a \neq 0$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy} = \lim_{(x,y)\to(0,0)} \frac{x^2+(ax)^2}{x(ax)} = \lim_{(x,y)\to(0,0)} \frac{(1+a^2)x^2}{ax^2} = \frac{1+a^2}{a},$$

while if $a = 0$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy} = \lim_{(x,y)\to(0,0)} \frac{x^2+(0x)^2}{x(0x)} = \lim_{(x,y)\to(0,0)} \frac{x^2}{0},$$

which is undefined.

(b) For $y = x^2$,

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy} = \lim_{(x,y)\to(0,0)} \frac{x^2+(x^2)^2}{x(x^2)} = \lim_{(x,y)\to(0,0)} \frac{1+x^2}{x},$$

which does not exist as the denominator approaches 0, but the numerator approaches 1.

(c) Since some paths do not have limit, $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy}$ does not exist.

4. (6 points) Verify the limit by definition:

$$\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2} = 0$$

Proof:

For any given $\varepsilon > 0$, we need to find δ in terms of ε, so that in the δ-neighborhood about $(0,0)$, whenever $0 < \sqrt{x^2+y^2} < \delta$, we always have $|f(x,y) - 0| < \varepsilon$.

Notice that

$$|f(x,y) - 0| = \left|\frac{5x^2y}{x^2+y^2}\right| = 5|y|\left(\frac{x^2}{x^2+y^2}\right) \leq 5|y| \leq 5\sqrt{x^2+y^2} < 5\delta.$$

So, if we take $\varepsilon = 5\delta$, that is, $\delta = \varepsilon/5$, then whenever $0 < \sqrt{x^2+y^2} < \delta$, we have $|f(x,y) - 0| < \varepsilon$.

It follows from the definition of limit that $f(x,y)$ approaches 0 as $(x,y) \to (0,0)$. QED (quod erat demonstrandum).