Quiz #3 Solution

Date: 02/12/2013 Name: ______________________

NOTE: You must show all work to earn credit.

1. (9 points) Given $\vec{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t\rangle$. Find (a) $\vec{r}'(t)$, (b) $\vec{r}''(t)$, and (c) $\vec{r}'(t) \cdot \vec{r}''(t)$.

Solution:

Given $\vec{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t\rangle$.

(a) $\vec{r}'(t) = \langle -\sin t + \sin t + t\cos t, \cos t - \cos t + t\sin t, 1\rangle = \langle t\cos t, t\sin t, 1\rangle$

(b) $\vec{r}''(t) = \langle \cos t - t\sin t, \sin t + t\cos t, 0\rangle$

(c) $\vec{r}'(t)\cdot\vec{r}''(t) = \langle t\cos t, t\sin t, 1\rangle \cdot \langle \cos t - t\sin t, \sin t + t\cos t, 0\rangle$
$= t\cos t\,(\cos t - t\sin t) + t\sin t\,(\sin t + t\cos t) + 0$
$= t\cos^2 t - t^2\cos t\sin t + t\sin^2 t + t^2\sin t\cos t = t(\cos^2 t + \sin^2 t) = t$

2. (6 points) Evaluate the definite integral:

$$\int_0^2 \left(t\vec{\imath} + e^t\vec{\jmath} - te^t\vec{k}\right)dt.$$

Solution:

$$\int_0^2 \left(t\vec{\imath} + e^t\vec{\jmath} - te^t\vec{k}\right)dt = \left[\frac{t^2}{2}\vec{\imath} + e^t\vec{\jmath} - (te^t - e^t)\vec{k}\right]_0^2$$

$= \left[2\vec{\imath} + e^2\vec{\jmath} - (2e^2 - e^2)\vec{k}\right] - \left[\vec{\jmath} + \vec{k}\right] = 2\vec{\imath} + (e^2 - 1)\vec{\jmath} + (-e^2 - 1)\vec{k}$.

3. (9 points) Find $\vec{r}(t)$ for the given conditions:
$\vec{r}'(t) = te^{-t^2}\vec{\imath} - e^{-t}\vec{\jmath} + \vec{k}$, and $\vec{r}(0) = \frac{1}{2}\vec{\imath} - \vec{\jmath} + \vec{k}$.

Solution:

$$\vec{r}(t) = \int \left(te^{-t^2}\vec{\imath} - e^{-t}\vec{\jmath} + \vec{k}\right)dt = -\frac{1}{2}e^{-t^2}\vec{\imath} + e^{-t}\vec{\jmath} + t\vec{k} + \vec{C}.$$

$$\vec{r}(0) = -\frac{1}{2}\vec{\imath} + \vec{\jmath} + 0\vec{k} + \vec{C} = \frac{1}{2}\vec{\imath} - \vec{\jmath} + \vec{k}.$$

Therefore, $\vec{C} = \vec{\imath} - 2\vec{\jmath} + \vec{k}$. It follows that

$$\vec{r}(t) = -\frac{1}{2}e^{-t^2}\vec{\imath} + e^{-t}\vec{\jmath} + t\vec{k} + \vec{C} = -\frac{1}{2}e^{-t^2}\vec{\imath} + e^{-t}\vec{\jmath} + t\vec{k} + \vec{\imath} - 2\vec{\jmath} + \vec{k},$$ or

$$\vec{r}(t) = \left(-\frac{1}{2}e^{-t^2} + 1\right)\vec{\imath} + (e^{-t} - 2)\vec{\jmath} + (t+1)\vec{k}.$$

4. (6 points) Evaluate the limit:

$$\lim_{t\to 0}\left(e^{-t}\vec{\imath} + \frac{1-\cos t}{t}\vec{\jmath} + \frac{\sin 3t}{t}\vec{k}\right)$$

Solution:

$$\lim_{t\to 0}\left(e^{-t}\vec{\imath} + \frac{1-\cos t}{t}\vec{\jmath} + \frac{\sin 3t}{t}\vec{k}\right) = \lim_{t\to 0} e^{-t}\vec{\imath} + \lim_{t\to 0}\frac{1-\cos t}{t}\vec{\jmath} + \lim_{t\to 0}\frac{\sin 3t}{t}\vec{k}$$

$$= \vec{\imath} + \lim_{t\to 0}\frac{\sin t}{1}\vec{\jmath} + \lim_{t\to 0}\frac{\sin 3t}{3t}3\vec{k} = \vec{\imath} + 0\vec{\jmath} + 3\vec{k} = \vec{\imath} + 3\vec{k}.$$