Lecture 35

15.6. Surface Integrals

Goals: (1) Evaluate a surface integral as a double integral.

(2) Evaluate a surface integral for a parametric surface.

(3) Determine the orientation of a surface.

(4) Understand the concept of a flux integral.

Questions:

- How to find a surface area (p. 1021)?
- What is the line integral formula (p. 1071)?

15.6.1. **Surface integrals for $\boldsymbol{z = g(x, y)}$**

(1) Formula:

Let S be a surface with equation $z = g(x, y)$ and let R be its projection onto the xy-plane. If g, g_x', g_y' are continuous on R and f is continuous on S, then the surface integral of f over S is

$$\iint_S f(x, y, z)dS = \iint_R f\big(x, y, g(x, y)\big)\sqrt{1 + [g_x'(x, y)]^2 + \left[g_y'(x, y)\right]^2}dA.$$

Important Note:

During evaluating the integrals, polar coordinates may be used.

(a) If S is a surface with equation $y = g(x, z)$, then we have

$$\iint_S f(x, y, z)dS = \iint_R f(x, g(x, z), z)\sqrt{1 + [g_x'(x, z)]^2 + [g_z'(x, z)]^2}dA$$

(b) If S is a surface with equation $x = g(y, z)$, then we have

$$\iint_S f(x, y, z)dS = \iint_R f(g(y, z), y, z)\sqrt{1 + \left[g_y'(y, z)\right]^2 + [g_z'(y, z)]^2}dA$$

(c) If $f(x, y, z) = 1$, then the formula becomes surface area (p. 1021):

$$\iint_S 1dS = \iint_R \sqrt{1 + \left[g_y'(y, z)\right]^2 + [g_z'(y, z)]^2}dA$$

(2) Examples 1, 2: evaluating surface integrals (pp. 1113-1114)

Usually, project the surface onto a coordinate plane.

Try exercises 1-6

(3) Example 3: finding the mass of a surface lamina (p. 1115)
Try exercises 11-12

15.6.2. **Surface integrals for $\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$**

(1) Formula:
Let S be a smooth parametric surface given by vector-valued function

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

defined over an open region D in the uv-plane. The surface integral of f over S is

$$\iint_S f(x,y,z)dS = \iint_D f\big(x(u,v), y(u,v), z(u,v)\big)\|\vec{r}_u'(u,v) \times \vec{r}_v'(u,v)\|dA$$

Important Note:
During evaluating the integrals, polar coordinates may be used.

(a) This is similar to the line integral formula (p. 1071):

$$\int_C f(x,y,z)ds = \int_a^b f\big(x(t), y(t), z(t)\big)\|\vec{r}'(t)\|dt$$

(b) When $f(x,y,z) = 1$, the formula becomes surface area (p. 1106):

$$\iint_S dS = \iint_D \|\vec{r}_u' \times \vec{r}_v'\|dA$$

(2) Example 4: evaluating a surface integral in parametric form (p. 1116)
Try exercises 13-16 (2-dim), 17-22 (3-dim).

15.6.3. **Orientation of surface**

We use normal vector to determine the orientation of a surface. Here the gradient of the surface plays an important role! In fact, the unit vector of the gradient is the upward unit normal vector, while the opposite unit vector of the gradient is the downward unit normal vector.

(1) Definition: An oriented surface has two distinct sides. See Figure 15.50.

(a) If $z = g(x,y)$, let $G(x,y,z) = z - g(x,y)$. The unit vector

$$\vec{N}_1 = \frac{\nabla G(x,y,z)}{\|\nabla G(x,y,z)\|} = \frac{-g_x'(x,y)\vec{i} - g_y'(x,y)\vec{j} + \vec{k}}{\sqrt{1 + [g_x'(x,y)]^2 + \left[g_y'(x,y)\right]^2}}$$

is called the **upward unit normal vector**. While

$$\vec{N}_2 = \frac{-\nabla G(x,y,z)}{\|\nabla G(x,y,z)\|} = \frac{g_x'(x,y)\vec{i} + g_y'(x,y)\vec{j} - \vec{k}}{\sqrt{1 + [g_x'(x,y)]^2 + \left[g_y'(x,y)\right]^2}}$$

is called the **downward unit normal vector**.

(b) If $\vec{r}(u,v) = x(u,v)\vec{\imath} + y(u,v)\vec{\jmath} + z(u,v)\vec{k}$, then the unit vector

$$\vec{N}_1 = \frac{\vec{r}_u' \times \vec{r}_v'}{\|\vec{r}_u' \times \vec{r}_v'\|}$$

is the **upward unit normal vector**. While

$$\vec{N}_2 = \frac{\vec{r}_v' \times \vec{r}_u'}{\|\vec{r}_v' \times \vec{r}_u'\|} = -\frac{\vec{r}_u' \times \vec{r}_v'}{\|\vec{r}_u' \times \vec{r}_v'\|}$$

is the **downward unit normal vector**.

(2) Examples of oriented surfaces

- Spheres
- Paraboloids
- Ellipsoids
- Planes

(3) Examples of non-oriented surfaces

- Möbius stripe (p. 1111)
- Klein bottle (p. 821)

15.6.4. **Flux integrals**

A primary application of surface integral is the flow of a fluid through a surface, which is the volume of fluid crossing the surface per unit of time, called the *flux of vector field across the surface*.

(1) Definition:

Let $\vec{F}(x,y,z) = M\vec{\imath} + N\vec{\jmath} + P\vec{k}$ be a vector field, where M, N, P have continuous first partial derivatives on the oriented surface S with the unit normal vector $\vec{N}_1$. The **flux integral of $\vec{F}$ across S** is given by

$$\iint_S \vec{F} \cdot \vec{N}_1 dS$$

Notice that $dS = \sqrt{1 + [g_x'(x,y)]^2 + [g_y'(x,y)]^2}\, dA$. Geometrically, a flux integral is the surface integral over S of the *normal component* of $\vec{F}$.

(2) Formulas:

(a) If S is a surface with equation $z = g(x,y)$ and R is its projection onto the xy-plane, then

$$\iint_S \vec{F} \cdot \vec{N}_1\, dS = \iint_R \vec{F} \cdot [\, -g_x'(x,y)\vec{\imath} - g_y'(x,y)\vec{\jmath} + \vec{k}\,]\, dA$$

$$\iint_S \vec{F} \cdot \vec{N}_2\, dS = \iint_R \vec{F} \cdot [\, g_x'(x,y)\vec{\imath} + g_y'(x,y)\vec{\jmath} - \vec{k}\,]\, dA$$

The surface is oriented upward in the first integral, while downward in the second one.

(b) If S is an oriented surface given by vector-valued function $\vec{r}(u,v) = x(u,v)\vec{\imath} + y(u,v)\vec{\jmath} + z(u,v)\vec{k}$ defined over a region D in the uv-plane, then

$$\iint_S \vec{F} \cdot \vec{N}_1 \, dS = \iint_D \vec{F} \cdot \left(r_u'(u,v) \times r_v'(u,v) \right) dA$$

(3) Example 5: finding the rate of mass flow using the flux integral (p. 1119)
Try exercises 23-28

15.6.5. **Homework Set #35**

- Read 15.6 (pages 1112-1121).
- Do exercises on pages 1122-1123:
  1, 3, 5, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29