Lecture 34

15.5. Parametric Surfaces

Goals: (1) Understand the definition of and sketch a parametric surface.

(2) Find a set of parametric equations to represent a surface.

(3) Find a normal vector and a tangent plane to a parametric surface.

(4) Find the area of a parametric surface.

Questions:

- How to represent a curve (in plane or space) parametrically?
- How to find normal vector of a curve?

15.5.1. **Sketching surfaces using a set of parametric equations**

A curve can be represented by using one parameter t:
$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, which can be called **parametric curve**. And

$$x = x(t), \qquad y = y(t), \qquad z = z(t)$$

can be called **parametric equations for the curve**. Notice that if the curve is in the plane, then $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$.
Similarly, a surface can be represented by using two parameters u, v.

(1) Definition:
Let x, y, and z be functions of u and v that are continuous on a domain D in the uv-plane. The set of points (x, y, z) given by

$$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$$

is called a **parametric surface**. The equations (components)

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v)$$

are called **parametric equations for the surface**. See Figure 15.35 for a visual example.
Note: the parametric representations of surfaces or curves are *not* unique. To identify a surface, sometimes we use trigonometric identities to eliminate the parameters. Try exercises 7-10.

(2) Examples 1, 2: sketching a parametric surface (p. 1103)
For fixed u, $\vec{r}(u, v)$ traces out latitude circles.
For fixed v, $\vec{r}(u, v)$ traces out longitude semi-circles as $0 \le u \le \pi$.
Try exercises 1-6, 17-20

15.5.2. **Finding parametric equations for surfaces**

(1) Example 3: representing a surface parametrically (p. 1104)
Try exercises 21-30

(2) Example 4: representing a surface of revolution parametrically (p.1104)
Try exercises 31--34

15.5.3. **Normal vectors and tangent planes**

(1) Normal vector formula:
Let S be a smooth parametric surface

$$\vec{r}(u,v) = x(u,v)\vec{\imath} + y(u,v)\vec{\jmath} + z(u,v)\vec{k}$$

defined over an open region D in the uv-plane. Let (u_0, v_0) be a point in D. A normal vector at the point

$$(x_0, y_0, z_0) = \left(x(u_0,v_0), y(u_0,v_0), z(u_0,v_0)\right)$$

is given by

$$\vec{N} = \vec{r}_u'(u_0,v_0) \times \vec{r}_v'(u_0,v_0) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}.$$

(2) Example 5: finding a tangent plane to a parametric surface using a normal vector (p. 1106)
Try exercises 31-34

15.5.4. **Area of a parametric surface**

(1) Surface area formula:
Let S be a smooth parametric surface

$$\vec{r}(u,v) = x(u,v)\vec{\imath} + y(u,v)\vec{\jmath} + z(u,v)\vec{k}$$

defined over an open region D in the uv-plane. If each point on the surface S corresponds to exactly one point in the domain D, then the **surface area** of S is given by

$$\text{Surface area} = \iint_S dS = \iint_D \|\vec{N}\| dA = \iint_D \|\vec{r}_u' \times \vec{r}_v'\| dA$$

where

$$\vec{r}_u' = \frac{\partial x}{\partial u}\vec{\imath} + \frac{\partial y}{\partial u}\vec{\jmath} + \frac{\partial z}{\partial u}\vec{k}, \text{ and } \vec{r}_v' = \frac{\partial x}{\partial v}\vec{\imath} + \frac{\partial y}{\partial v}\vec{\jmath} + \frac{\partial z}{\partial v}\vec{k}.$$

(2) Example 6: finding surface area (p. 1107)
Try exercises 39-46

15.5.5. **Homework Set #34**

- Read 15.5 (pages 1102-1108).
- Do exercises on pages 1109-1111:
  1-6, 7, 13, 15, 19, 21, 23, 25, 27, 29, 31, 33, 39, 41