Lecture 32

15.3. Conservative Vector Fields and Independence of Path

Goals: (1) Understand and use the Fundamental Theorem of Line Integrals.

(2) Understand the concept of independence of path.

(3) Understand the concept of conservation of energy.

Questions:

- What is a conservative vector field?
- What is the kinetic energy?

15.3.1. **Fundamental Theorem of Line Integrals (FTLI)**

Recall: By the Chain rule,

$$\frac{d}{dt}f\big(x(t),y(t)\big) = f_x'(x,y)\frac{dx}{dt} + f_y'(x,y)\frac{dy}{dt}.$$

Or

$$df\big(x(t),y(t)\big) = f_x'(x,y)dx + f_y'(x,y)dy.$$

(1) Fundamental Theorem of Line Integrals with conservative vector fields:
Let C be a piecewise smooth curve lying in an open region R and given by

$$\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath}, \qquad a \le t \le b.$$

If $\vec{F}(x,y) = M\vec{\imath} + N\vec{\jmath}$ is *conservative* in R, and M and N are continuous in R, then

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f\big(x(b),y(b)\big) - f\big(x(a),y(a)\big)$$

where f is a potential function of $\vec{F}$. That is, $\vec{F}(x,y) = \nabla f(x,y)$.

(2) Example 1: all paths lead to the same integral value (p. 1083)
Try exercises 1-4, 5-10, 11-24
Examples 2, 3: using the FTLI (pp. 1085-1086)
Try exercises 25-34

15.3.2. **Independence of path**

(1) First, if the vector field is conservative, then you can pick a special path to evaluate the line integral. Secondly, sometimes you may need to verify whether the vector field is conservative or not. Lastly, if the vector field is conservative, then the line integral between any two points is simply the difference in the values of the potential function at those two points. That is, it is independent of path!

(2) Path independence vs. Conservative
If $\vec{F}$ is *continuous* on an open *connected* region, then the line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

is independent of path if and only if $\vec{F}$ is conservative.
(3) Example 4: finding work in a conservative force field (p. 1087)
Try exercises 35-40
(4) Closed curve case:
Let $\vec{F}(x, y, z) = M\vec{i} + N\vec{j} + P\vec{k}$ have continuous first partial derivatives in an open connected region R, and let C be a piecewise smooth curve in R. The following conditions are equivalent:
(a) $\vec{F}$ is conservative.
(b) $\int_C \vec{F}(x, y, z) \cdot d\vec{r}$ is independent of path.
(c) $\int_C \vec{F}(x, y, z) \cdot d\vec{r} = 0$ for every closed curve C in R.
(5) Example 5: evaluating a line integral along a closed path (p. 1088)

15.3.3. **Conservation of Energy**

(1) Definition:
Let $\vec{F}$ be a *conservative* vector field defined on a smooth curve C. The **kinetic energy** of a particle of mass m and speed v is $k = \frac{1}{2}mv^2$. The **potential energy** p of a particle at point (x, y, z) in a conservative vector field $\vec{F}$ is defined as $p(x, y, z) = -f(x, y, z)$, where f is the potential function of $\vec{F}$.
(2) Law of Conservation of Energy:
The sum of the potential and kinetic energies remains constant from point to point:

$$p(A) + k(A) = p(B) + k(B).$$

15.3.4. **Homework Set #32**

- Read 15.3 (pages 1083-1089).
- Do exercises on pages 1090-1092:
  1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 35, 37, 47-50