Lecture 31

15.2. Line Integrals

Goals: (1) Understand and use the concept of a piecewise smooth curve.
(2) Write and evaluate a line integral.
(3) Write and evaluate a line integral of a vector field.
(4) Write and evaluate a line integral in differential form.

Question:

- What is the definition of smooth vector-valued function?

15.2.1. **Piecewise smooth curves**

Recall: In chapter 12, we have vector-valued functions $\vec{r}(t) = f(t)\vec{\imath} + g(t)\vec{\jmath}$ for plane, and $\vec{r}(t) = f(t)\vec{\imath} + g(t)\vec{\jmath} + h(t)\vec{k}$ for space. $\vec{r}(t)$ is smooth if the derivatives of components are continuous and $\vec{r}'(t) \neq \vec{0}$.

(1) Definition: A curve C is **piecewise smooth** if the interval $[a, b]$ can be partitioned into a finite number of subintervals, on each of which C is smooth.
(2) Example 1: finding a piecewise smooth parametrization (p. 1069)
Try exercises 1-6

15.2.2. **Line integrals**

(1) Definition:
If f is defined in a region containing a smooth curve C of finite length, then the line integral of f along C is given by

$$\int_C f(x, y)\, ds = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta s_i \quad \text{plane}$$

$$\int_C f(x, y, z)\, ds = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta s_i \quad \text{space}$$

provided the limit exists.

(2) How to evaluate line integral?
Let f be continuous in a region containing a smooth curve C. If C is given by $\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath}$ in plane, or $\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$ in space, where $a \leq t \leq b$, then $ds = \|\vec{r}'(t)\|dt$. Hence the formulas for line integrals are:

$$\int_C f(x,y)\,ds = \int_a^b f\big(x(t),y(t)\big)\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt$$

$$\int_C f(x,y,z)\,ds = \int_a^b f\big(x(t),y(t),z(t)\big)\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt$$

<u>Note</u>: If $f = 1$, then we get the arc length of the curve C.
If C is a piecewise smooth path composed of smooth curves $C_1, C_2, \dots, C_n$, then

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \cdots + \int_{C_n} f\,ds$$

(3) Example 2: evaluating a line integral over a line segment (p. 1071)
Try exercises 7-10
(4) Examples 3, 4: evaluating a line integral over a path (pp. 1072-1073)
Try exercises 11-20
(5) Example 5: evaluating the mass of spring (p. 1073)
Try exercises 21-26

15.2.3. **Line integrals of a vector field**

<u>Recall</u>: work = force × distance. On p. 789,

$$\Delta W_i = \left[\vec{F}(x_i,y_i,z_i)\cdot\vec{T}(x_i,y_i,z_i)\right]\Delta s_i$$

Therefore,

$$W = \int_C \vec{F}(x,y,z)\cdot\vec{T}(x,y,z)\,ds$$

(1) <u>Definition</u>:

Let $\vec{F}$ be a continuous vector field (also called **force field**) defined on a smooth curve C given by $\vec{r}(t), a \le t \le b$. The **line integral** of vector field $\vec{F}$ on C is given by

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}ds = \int_a^b \vec{F}\big(x(t),y(t),z(t)\big)\cdot\vec{r}'(t)\,dt$$

(2) Example 6: finding work done by a force (p. 1075)
Try exercises 27-32
(3) Example 7: finding work done through an oriented path (p. 1076)
Try exercises 35-40

15.2.4. **Line integrals in differential form**

(1) <u>Definition</u>:
The **differential form** of a line integral of a vector field (in plane) $\vec{F}(x,y) = M\vec{i} + N\vec{j}$ along a curve C given by $\vec{r}(t), a \le t \le b$ is:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C Mdx + Ndy$$

The **differential form** of a line integral of a vector field (in space) $\vec{F}(x,y,z) = M\vec{i} + N\vec{j} + P\vec{k}$ along a curve C given by $\vec{r}(t), a \le t \le b$ is:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C Mdx + Ndy + Pdz$$

(2) Examples 8, 9: finding a line integral in differential form (pp. 1077-1078)
Try exercises 45-50, 51-62

15.2.5. **Homework Set #31**

- Read 15.2 (pages 1069-1078).
- Do exercises on pages 1079-1082:
  3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 35, 37, 43, 45, 47, 49, 51, 53, 55, 57, 59, 61, 65, 69, 83-86