Lecture 30

15.1. Vector Fields

Goals: (1) Understand the concept of a vector field.

(2) Determine whether a vector field is conservative.

(3) Find the curl of a vector field.

(4) Find the divergence of a vector field.

Questions:

- What are vectors in 2-dim plane and 3-dim space?
- What is a vector-valued function discussed in chapter 12?

15.1.1. **Vector fields**

Recall: In chapter 12, we have vector-valued functions $\vec{r}(t) = f(t)\vec{\imath} + g(t)\vec{\jmath}$ for plane, and $\vec{r}(t) = f(t)\vec{\imath} + g(t)\vec{\jmath} + h(t)\vec{k}$ for space.

(1) A vector field in plane:
Definition: Let M and N be functions of two variables x and y, defined on a plane region R. Then $\vec{F}(x, y) = M\vec{\imath} + N\vec{\jmath}$ is called a **vector field over R**.

(2) A vector filed in space:
Definition: Let M, N and P be functions of three variables x, y and z, defined on a space region Q. Then $\vec{F}(x, y, z) = M\vec{\imath} + N\vec{\jmath} + P\vec{k}$ is called a **vector field over Q**.

(3) The gradient of a regular function is a vector field!

$$\nabla f(x, y) = f_x'(x, y)\vec{\imath} + f_y'(x, y)\vec{\jmath}$$

$$\nabla f(x, y, z) = f_x'(x, y, z)\vec{\imath} + f_y'(x, y, z)\vec{\jmath} + f_z'(x, y, z)\vec{k}$$

(4) A vector field is continuous at a point if each of its component functions M, N and P is continuous at that point. Some common physical examples of vector fields are *velocity fields*, *gravitational fields*, and *electric force fields*.

(5) Definition: Let $\vec{r}(t) = f(t)\vec{\imath} + g(t)\vec{\jmath} + h(t)\vec{k}$ be a position vector. The vector field $\vec{F}$ is an **inverse square field** if

$$\vec{F}(x, y, z) = \frac{k}{||\vec{r}||^2}\vec{u}$$

where k is a real number and $\vec{u} = \vec{r}/||\vec{r}||$ is a unit vector in the direction of $\vec{r}$.

(6) Examples 1, 2: sketching a vector field in plane (p. 1060)
Try exercises 1-14

(7) Example 3: sketching a vector field in space (p. 1061)
Try exercises 15-16

15.1.2. **Conservative vector fields**

(1) Definition:
A vector field $\vec{F}$ is called **conservative** if there exists a differentiable function $\vec{F} = \nabla f$. The function f is called the **potential function** for $\vec{F}$.

(2) Example 4: finding conservative vector fields (p. 1061)
Try exercises 21-30

(3) Test for conservative vector field in the *plane*:
Let M and N have continuous first partial derivatives on an open disk R.
The vector field given by $\vec{F}(x,y) = M\vec{\imath} + N\vec{\jmath}$ is conservative if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Note: It tells us whether a vector field is conservative, but it does not show how to find the potential function.

(4) Example 5: testing conservative vector field in the *plane* (p. 1062)
Try exercises 31-34, 35-38

(5) Example 6: finding a potential function (p. 1063)
Try exercises 39-48

15.1.3. **Curl of a vector field in space**

Recall: In 13.6, we defined (p. 941)

$$\nabla f(x,y,z) = f_x(x,y,z)\vec{\imath} + f_y(x,y,z)\vec{\jmath} + f_z(x,y,z)\vec{k}.$$

Hence we can denote the "differential operator" ∇ as follows:

$$\nabla = \left(\frac{\partial}{\partial x}\right)\vec{\imath} + \left(\frac{\partial}{\partial y}\right)\vec{\jmath} + \left(\frac{\partial}{\partial z}\right)\vec{k} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$

Similarly, we can define another "differential operator" ∇^2 as follows:

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2}\right)\vec{\imath} + \left(\frac{\partial^2}{\partial y^2}\right)\vec{\jmath} + \left(\frac{\partial^2}{\partial z^2}\right)\vec{k}$$

(1) Definition:
The **curl** of vector field $\vec{F}(x,y,z) = M\vec{\imath} + N\vec{\jmath} + P\vec{k}$ is

$$\overrightarrow{\text{curl}}\,\vec{F}(x,y,z) = \nabla \times \vec{F}(x,y,z) =$$
$$\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{\imath} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}.$$

If $\overrightarrow{\text{curl}}\,\vec{F}(x,y,z) = \vec{0}$, then $\vec{F}(x,y,z)$ is said to be **irrotational**.
Important Note: The curl of a vector field is introduced and used to test for conservative vector field *in space*! The curl of a vector field (in space) is also a vector field!

(2) Test for conservative vector field in *space*:
Let M, N and P have continuous first partial derivatives on an open sphere Q in space. The vector field given by $\vec{F}(x,y,z) = M\vec{\imath} + N\vec{\jmath} + P\vec{k}$ is

conservative if and only if $\overrightarrow{\text{curl}}\,\vec{F}(x,y,z)=\vec{0}$. That is,

$$\frac{\partial P}{\partial y}=\frac{\partial N}{\partial z},\qquad \frac{\partial M}{\partial z}=\frac{\partial P}{\partial x},\qquad \frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}.$$

(3) Example 7: finding curl of vector field and testing conservative vector field in *space* (p. 1064)
Try exercises 49-52

(4) Example 8: finding potential function for vector field in *space* (p. 1065)
Try exercises 51-56

15.1.4. **Divergence of a vector field**

(1) Definition:
The **divergence** of a vector field (in plane) $\vec{F}(x,y)=M\vec{\imath}+N\vec{\jmath}$ is:

$$\text{div}\,\vec{F}(x,y)=\nabla\cdot\vec{F}(x,y)=\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}.$$

The **divergence** of a vector field (in space) $\vec{F}(x,y,z)=M\vec{\imath}+N\vec{\jmath}+P\vec{k}$ is:

$$\text{div}\,\vec{F}(x,y,z)=\nabla\cdot\vec{F}(x,y,z)=\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}.$$

If $\text{div}\,\vec{F}=0$, that is, the sum of partial derivatives equals 0, then $\vec{F}$ is said to be **divergence free**.
Note: The divergence of a vector field is a scalar, *not* a vector field!

(2) Example 9: finding the divergence of a vector field (p. 1066)
Try exercises 57-62, 63-66, 67-70

15.1.5. **Properties of curl and divergence of vector field**

Let $\vec{F}$ and $\vec{G}$ be vector fields and their component functions have continuous second partial derivatives. Let f be scalar function. (see Exercises 83-90)

(1) $\overrightarrow{\text{curl}}\,(\vec{F}+\vec{G})=\overrightarrow{\text{curl}}\,\vec{F}+\overrightarrow{\text{curl}}\,\vec{G}$

(2) $\overrightarrow{\text{curl}}\,(\nabla f)=\nabla\times(\nabla f)=\vec{0}$

(3) $\text{div}\,(\vec{F}+\vec{G})=\text{div}\,\vec{F}+\text{div}\,\vec{G}$

(4) $\text{div}\,(\vec{F}\times\vec{G})=(\overrightarrow{\text{curl}}\,\vec{F})\cdot\vec{G}-\vec{F}\cdot(\overrightarrow{\text{curl}}\,\vec{G})$

(5) $\nabla\times[\nabla f+(\nabla\times\vec{F})]=\nabla\times(\nabla\times\vec{F})$

(6) $\nabla\times(f\vec{F})=f(\nabla\times\vec{F})+(\nabla f)\times\vec{F}$

(7) $\text{div}\,(f\vec{F})=f\,\text{div}\,\vec{F}+\nabla f\cdot\vec{F}$

(8) $\text{div}\,(\overrightarrow{\text{curl}}\,\vec{F})=0$

15.1.6. **Homework Set #30**

- Read 15.1 (pages 1058-1066).
- Do exercises on pages 1067-1068:
  1, 3, 5, 9, 11, 13, 15, 21, 23, 25, 27, 29, 31, 35, 37, 43, 45, 49, 51, 57, 59, 61, 63, 65, 67, 69, 75, 95-98