Lecture 28

14.6. Triple Integrals and Applications

Goals: (1) Use a triple integral to find the volume of a solid region.

(2) Find the center of mass and moments of inertia of a solid region.

Questions:

- What is Fubini's Theorem for double integrals?
- How to find the center of mass for planar region?

14.6.1. **Triple integrals**

(1) Definition of triple integral:

If f is a continuous function over a bounded solid region Q, then the **triple integral of f over Q** is defined as

$$\iiint_Q f(x,y,z)dV = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i,y_i,z_i)\Delta V_i$$

provided the limit exists.

In particular, the volume of the solid region Q is given by

$$\text{Volume of } Q = \iiint_Q dV$$

(2) Properties of triple integrals

1. $\iiint_Q cf(x,y,z)dV = c\iiint_Q f(x,y,z)dV$

2. $\iiint_Q [f(x,y,z) \pm g(x,y,z)]dV = \iiint_Q f(x,y,z)dV \pm \iiint_Q g(x,y,z)dV$

3. $\iiint_Q f(x,y,z)dV = \iiint_{Q_1} f(x,y,z)dV + \iiint_{Q_2} f(x,y,z)dV$

where Q is union of two non-overlapping solid subregions Q_1 and Q_2.

(3) Fubini's Theorem on triple integrals

Let f be continuous on a solid region Q defined by

$$a \le x \le b, \qquad h_1(x) \le y \le h_2(x), \qquad g_1(x,y) \le z \le g_2(x,y)$$

where $h_1, h_2, g_1,$ and g_2 are continuous functions. Then,

$$\iiint_Q f(x,y,z)dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)dz\,dy\,dx.$$

Note: there are 5 more similar iterated triple integral formulas. Can you list all of them?

(4) Example 1: evaluating triple integral (p. 1028)
Try exercises 1-8

(5) Example 2: finding volume by using triple integral (p. 1029)
Try exercises 23-26

(6) Example 3: changing the order of integration (p. 1030)
Try exercises 27-32

(7) Example 4: determining the limits of integration (p. 1031)
Try exercises 13-18

14.6.2. **Center of mass**

(1) Definition of mass:
If ρ is a continuous density function on a solid region Q, then the mass m of the solid is given by

$$m = \iiint_Q \rho(x,y,z)dV$$

Note: this is the same formula for triple integral.

(2) Define three moments of mass by triple integrals:

$$M_{yz} = \iiint_Q x\rho(x,y,z)dV$$

$$M_{zx} = \iiint_Q y\rho(x,y,z)dV$$

$$M_{xy} = \iiint_Q z\rho(x,y,z)dV$$

(3) The center of mass is given by

$$\bar{x} = \frac{M_{yz}}{m}, \qquad \bar{y} = \frac{M_{zx}}{m}, \qquad \bar{z} = \frac{M_{xy}}{m}.$$

Note: if the density function is constant, then the center of mass is also called the **centroid**.

(4) Example 5: finding the center of mass for solid region (p. 1033)
Try exercises 39-44.

14.6.3. **Moments of inertia**

(1) First moments.
The moments of mass M_{yz}, M_{zx}, and M_{xy} are also called the **first moments** about yz-plane, zx-plane, and xy-plane, respectively.

(2) Second moments are also called the **moments of inertia** of a solid region about x-axis, y-axis, and z-axis, respectively:

$$I_x = \iiint_Q (y^2 + z^2)\rho(x,y,z)dV$$

$$I_y = \iiint_Q (z^2 + x^2)\rho(x,y,z)dV$$

$$I_z = \iiint_Q (x^2 + y^2)\rho(x,y,z)dV$$

Note: usually we find three intermediate triple integrals in order to find the above moments of inertia:

$$I_{yz} = \iiint_Q x^2\rho(x,y,z)dV$$

$$I_{zx} = \iiint_Q y^2\rho(x,y,z)dV$$

$$I_{xy} = \iiint_Q z^2\rho(x,y,z)dV$$

Then $I_x = I_{zx} + I_{xy}$, $I_y = I_{xy} + I_{yz}$, and $I_z = I_{yz} + I_{zx}$.

(3) Example 6: finding the moments of inertia (p. 1034)
Try exercises 55-58

14.6.4. **Homework Set #28**

- Read 14.6 (pages 1027-1034).
- Do exercises on pages 1035-1037:
  1, 3, 5, 7, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 61, 63