Lecture 26

14.4. Center of Mass and Moments of Inertia

Goals: (1) Find the mass of a planar lamina using a double integral.

(2) Find the center of mass of a planar lamina using double integrals.

(3) Find moments of inertia using double integrals.

Questions:

- How to find the mass of a lamina?
  A: p. 502
- What is a planar lamina and how to find its center of mass?
  A: p. 502

14.4.1. **Mass**

Density is normally expressed as mass per unit volume for 3-dim or mass per unit surface area for 2-dim. The definition of *lamina* will be extended to include thin plates of *variable* density. Double integrals will be used to find the mass of a lamina of variable density, where the density at (x, y) is given by the **density function** $\rho(x, y)$.

(1) Definition of mass of a planar lamina of variable density:
If ρ is a continuous density function on the lamina corresponding to a plane region R, then the mass m of the lamina is given by

$$m = \iint_R \rho(x, y)dA.$$

Note: this is the same formula for volume (the function is height instead of density)!

(2) Example 1: use rectangular coordinates for vertically or horizontally simple regions (p. 1012)

(3) Example 2: use polar coordinates for r-simple or θ-simple regions (p. 1013)
Try exercises 1-4.

14.4.2. **Moments of mass and center of mass**

(1) Definition of moments of mass (See Figure 14.37):
Let ρ be a continuous density function on the planar lamina region R. The **moments of mass** m with respect to the x- and y-axes are

$$M_x = \iint_R y\rho(x,y)dA \text{ and } M_y = \iint_R x\rho(x,y)dA.$$

(2) Definition of center of mass
Given the mass m of the lamina, the moments of mass M_x and M_y. Then the **center of mass** is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

If the region is a simple plane region rather than a lamina, the center of mass will be called the **centroid** of the region.

(3) Example 3: finding the center of mass for polar region (p. 1015)
Try exercises 5-8, 11-22.

14.4.3. **Moments of inertia**

(1) First moments.
The moments of mass M_x and M_y are also called the **first moments** about the x-axis and y-axis.

(2) Second moments are also called the **moment of inertia** of a lamina about a line:

$$I_x = \iint_R y^2\rho(x,y)dA \text{ and } I_y = \iint_R x^2\rho(x,y)dA$$

The sum of I_x and I_y is called the **polar moment of inertia**, denoted by I_0.

$$I_0 = I_x + I_y = \iint_R (x^2 + y^2)\rho(x,y)dA$$

Similarly, the **radii of gyration** about the y-axis and x-axis are:

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}}, \qquad \bar{\bar{y}} = \sqrt{\frac{I_x}{m}}$$

(3) Example 4: finding the moment of inertia (p. 1016)
Try exercises 33-40

14.4.4. **Homework Set #26**

- Read 14.4 (pages 1012-1017).
- Do exercises on pages 1018-1019:
  3, 5, 7, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31