# Lecture 25

## 14.3. Change of Variables: Polar Coordinates

**Goals:** (1) Write and evaluate double integrals in polar coordinates.

Questions:

- How to find the arc length if radius is $r$ and angle is $\theta$?
- What is the relationship between polar coordinates and rectangular coordinates?

14.3.1. **Review of polar coordinates**

Some double integrals are *much* easier to evaluate in polar form than in rectangular form. This is especially true for regions such as circles, sectors, cardioids, and rose curves, and for integrands that involve $x^2 + y^2$. Of course, if the region is vertically or horizontally simple, then it will be easier to use rectangular form than in polar form. So, it is important to understand why we need to learn this section and when we should use polar form.

(1) The polar coordinates $(r, \theta)$of a point are related to the rectangular coordinates $(x, y)$ of the point as follows:

$$x = r\cos\theta \text{ and } y = r\sin\theta.$$
$$r^2 = x^2 + y^2 \text{ and } \tan\theta = \frac{y}{x}.$$

(2) Polar sector is a region that can be described using polar coordinates as follows (see Figure 14.25):

$$R = \{(x, y) \mid r_1 \le r \le r_2, \theta_1 \le \theta \le \theta_2\}$$

(3) Example 1 (p. 1004)
Try exercises 1-8.

14.3.2. **Evaluating double integral by polar coordinates**

(1) In addition to vertically and horizontally simple regions, there are two more special regions.

- $\boldsymbol{r}$**-simple** region: fixed bounds for $\theta$, but variable bounds for $r$.

$$\alpha \le \theta \le \beta, \qquad 0 \le g_1(\theta) \le r \le g_2(\theta)$$

See Figure 14.29

- $\boldsymbol{\theta}$**-simple** region: fixed bounds for $r$, but variable bounds for $\theta$.

$$r_1 \le r \le r_2, \qquad 0 \le h_1(r) \le \theta \le g_2(r)$$

See Figure 14.29

(2) Theorem 14.3: Let $R$ be a plane region consisting of all points $(x, y) = (r\cos\theta, r\sin\theta)$ satisfying the conditions $0 \le g_1(\theta) \le r \le g_2(\theta), \alpha \le \theta \le \beta$ (that is, $r$-simple), where $0 \le (\beta - \alpha) \le 2\pi$. If $g_1$ and $g_2$ are continuous on $[\alpha, \beta]$ and $f$ is continuous on $R$, then

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) r dr\, d\theta.$$

Note: See Figure 14.29: the bounds for $r$ are $g_1(\theta)$ and $g_2(\theta)$. When they are constant, then we get a polar sector.

(3) Example 2: evaluating double polar integral (p. 1006)
Try exercises 9-16

(4) Example 3: evaluating double integral by converting rectangular to polar coordinates (p. 1007)
Try exercises 17-26

(5) Example 4: using symmetry and polar coordinates to evaluate double integral (p. 1008)
Try exercises 27-32

(6) Example 5: evaluating double integral by different order (p. 1008)
Try exercises 37-42

(7) Others: Exercises 33-38, 43-48, 49-54

14.3.3. **Homework Set #25**

- Read 14.3 (pages 1004-1008).
- Do exercises on pages 1009-1011:
  3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 45, 47, 51, 53, 67, 68