Lecture 24

14.2. Double Integrals and Volume

Goals: (1) Use a double integral to represent the volume of a solid region.

(2) Use properties of double integrals.

(3) Evaluate a double integral as an iterated integral.

(4) Find the average value of a function over a region.

Questions:

- What is Wallis' formula?
  Answer: See p. 536
- How to approximate the area of a region in plane?
  Answer: Riemann sum

14.2.1. **Double integrals and volume of a solid region**

(1) Riemann sum in three dimensional region:

- Assume that $z = f(x,y) \geq 0$ is a continuous function, whose graph is a surface above a closed, bounded region R. Then the Riemann sum is used to approximate the volume of the solid region, given as follows:

$$\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$

where ΔA_i represents the area of ith rectangle, and $f(x_i, y_i)\Delta A_i$ is the volume of the ith prism. See Figure 14.11 (p. 992).

- Example 1 (p. 993)
  Try exercises 1-4.

(2) Definition of double integral:

- If f is defined on a closed, bounded region R in the xy-plane, then the ***double integral of f over R*** is given by

$$\iint_R f(x,y)\,dA = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$

provided the limit exists. If the limit exists, then f is integrable over R.

- Note: the usual definite integral is also called a ***single*** integral.

(3) Volume of a solid region:

- If $f(x,y) \geq 0$ is integrable on a closed, bounded region R in the xy-plane, then the volume of the solid region that lies above R and below the graph of f is defined as the double integral:

$$V = \iint_R f(x,y)dA.$$

- Geometrically, if the function is above the xy-plane, the double integral gives the volume of the solid region.

14.2.2. **Properties of double integrals (Theorem 14.1)**

Let f, g be continuous functions over a closed, bounded region R, and let c be a constant.

(1) Constant multiplication

$$\iint_R cf(x,y)dA = c\iint_R f(x,y)dA$$

(2) Sum/Difference

$$\iint_R [f(x,y) \pm g(x,y)]dA = \iint_R f(x,y)dA \pm \iint_R g(x,y)dA$$

(3) Positive

$$\iint_R f(x,y)dA \geq 0, \text{ if } f(x,y) \geq 0$$

(4) Inequality preserving

$$\iint_R f(x,y)dA \geq \iint_R g(x,y)dA\text{, if } f(x,y) \geq g(x,y)$$

(5) Union of two non-overlapping subregions

$$\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA\text{, where } R = R_1 \cup R_2$$

14.2.3. **Evaluation of double integrals**

According to Fubini's theorem, if region R is either vertically simple, or horizontally simple, then the double integral of f on R can be rewritten an iterated integral.

(1) Vertically simple region: if R is defined by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1, g_2 are continous on $[a, b]$, then

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dy\,dx$$

(2) Horizontally simple region: if R is defined by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where h_1, h_2 are continous on $[c, d]$, then

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dx\,dy$$

(3) Note: In section 14.1, we can consider the area to be the volume with constant height $z = f(x, y) = 1$.
(4) Example 2 (p. 996)
Try exercises 13-20, 21-30.
(5) Example 3 (p. 997) – paraboloid.
Try exercises 33-40.
(6) Example 4 (p. 998) – choosing the right order is crucial.
Try exercises 53-58.
(7) Example 5 (p. 999) – region bounded by two surfaces.
Try exercises 47-52.

14.2.4. **Average value of a function**

- Definition: If f is integrable over the plane region R, then the **average value** of f over R is

$$\frac{1}{A}\iint\limits_R f(x,y)dA$$

where A is the area of R.
- Example 6 (p. 1000)
Try exercises 59-64.

14.2.5. **Homework Set #24**

- Read 14.2 (pages 992-1000).
- Do exercises on pages 1000-1003:
1, 3, 7, 9, 13, 15, 17, 19, 21, 23, 25, 27, 33, 35, 37, 41, 43, 45, 49, 53, 55, 57, 61, 63, 85, 86