## Lecture 24

# 14.2. Double Integrals and Volume

Goals: (1) Use a double integral to represent the volume of a solid region.

- (2) Use properties of double integrals.
- (3) Evaluate a double integral as an iterated integral.
- (4) Find the average value of a function over a region.

#### Questions:

• What is Wallis' formula? Answer: See p. 536

How to approximate the area of a region in plane?
 Answer: Riemann sum

#### 14.2.1. Double integrals and volume of a solid region

- (1) Riemann sum in three dimensional region:
  - Assume that  $z = f(x, y) \ge 0$  is a continuous function, whose graph is a surface above a closed, bounded region R. Then the Riemann sum is used to approximate the volume of the solid region, given as follows:

$$\sum_{i=1}^{n} f(x_i, y_i) \Delta A_i$$

where  $\Delta A_i$  represents the area of *i*th rectangle, and  $f(x_i, y_i)\Delta A_i$  is the volume of the *i*th prism. See Figure 14.11 (p. 992).

- Example 1 (p. 993) Try exercises 1-4.
- (2) Definition of double integral:
  - If f is defined on a closed, bounded region R in the xy-plane, then the double integral of f over R is given by

$$\iint\limits_{\mathbb{R}} f(x,y)dA = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i$$

provided the limit exists. If the limit exists, then f is integrable over R.

- Note: the usual definite integral is also called a *single* integral.
- (3) Volume of a solid region:
  - If  $f(x, y) \ge 0$  is integrable on a closed, bounded region R in the xyplane, then the volume of the solid region that lies above R and below
    the graph of f is defined as the double integral:

$$V = \iint\limits_R f(x,y) dA.$$

• Geometrically, if the function is above the *xy*-plane, the double integral gives the volume of the solid region.

#### 14.2.2. Properties of double integrals (Theorem 14.1)

Let f, g be continuous functions over a closed, bounded region R, and let c be a constant.

(1) Constant multiplication

$$\iint\limits_{R} cf(x,y)dA = c\iint\limits_{R} f(x,y)dA$$

(2) Sum/Difference

$$\iint\limits_R [f(x,y) \pm g(x,y)] dA = \iint\limits_R f(x,y) dA \pm \iint\limits_R g(x,y) dA$$

(3) Positive

$$\iint\limits_{\mathbb{R}} f(x,y)dA \ge 0, \text{ if } f(x,y) \ge 0$$

(4) Inequality preserving

$$\iint f(x,y)dA \ge \iint g(x,y)dA, \text{ if } f(x,y) \ge g(x,y)$$

(5) Union of two non-overlapping subregions

$$\iint\limits_R f(x,y)dA = \iint\limits_{R_1} f(x,y)dA + \iint\limits_{R_2} f(x,y)dA, \text{ where } R = R_1 \cup R_2$$

#### 14.2.3. Evaluation of double integrals

According to Fubini's theorem, if region R is either vertically simple, or horizontally simple, then the double integral of f on R can be rewritten an iterated integral.

(1) Vertically simple region: if R is defined by  $a \le x \le b$  and  $g_1(x) \le y \le g_2(x)$ , where  $g_1, g_2$  are continuous on [a, b], then

$$\iint\limits_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dy dx$$

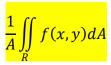
(2) Horizontally simple region: if R is defined by  $c \le y \le d$  and  $h_1(y) \le x \le h_2(y)$ , where  $h_1, h_2$  are continous on [c, d], then

$$\iint\limits_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dx \, dy$$

- (3) Note: In section 14.1, we can consider the area to be the volume with constant height z = f(x, y) = 1.
- (4) Example 2 (p. 996) Try exercises 13-20, 21-30.
- (5) Example 3 (p. 997) paraboloid. Try exercises 33-40.
- (6) Example 4 (p. 998) choosing the right order is crucial. Try exercises 53-58.
- (7) Example 5 (p. 999) region bounded by two surfaces. Try exercises 47-52.

### 14.2.4. Average value of a function

• Definition: If f is integrable over the plane region R, then the **average** value of f over R is



where A is the area of R.

• Example 6 (p. 1000) Try exercises 59-64.

#### 14.2.5. Homework Set #24

- Read 14.2 (pages 992-1000).
- Do exercises on pages 1000-1003: 1, 3, 7, 9, 13, 15, 17, 19, 21, 23, 25, 27, 33, 35, 37, 41, 43, 45, 49, 53, 55, 57, 61, 63, 85, 86