Lecture 22

13.10. Lagrange Multipliers

Goals: (1) Understand the Method of Lagrange Multipliers.

(2) Use Lagrange multipliers to solve constrained optimization problems.

(3) Use the Method of Lagrange Multipliers with two constraints.

Questions:

- How to find relative extrema of a function?
- What does Theorem 13.12 say about gradient and level curves (p. 940)?
- What does Theorem 13.14 say about gradient and level surfaces (p. 950)?

13.10.1. **Lagrange multipliers with one constraint**

(1) Background:
Usually in this kind of optimization problem, we are given a condition, $g(x,y)=c$, called ***constraint***, and we must determine an unknown value, which we need to set up a function (sometimes it's given also), say, $V=f(x,y)$ to do. Once setting up the function (also called ***objective function***), we find the gradients of both functions. When these gradients satisfy the condition that $\nabla f(a,b)=\lambda\nabla g(a,b)$, where λ is a scalar (which is called a ***Lagrange multiplier***, or ***marginal productivity of money*** in economics, which means that for each additional dollar spent on production, an additional λ unit of the product can be produced), then the objective function $V=f(x,y)$ will have an extremum at (a,b). For example, if we need to find the rectangle of maximum area that can be inscribed in the given ellipse: $\frac{x^2}{3^2}+\frac{y^2}{4^2}=1$, then let (x,y) be the vertex of the rectangle in the first quadrant, and we can set up the area function $A=4xy$. In this example, $g(x,y)=\frac{x^2}{3^2}+\frac{y^2}{4^2}=1$ (here $c=1$) is the constraint, and $f(x,y)=4xy$ is the objective function. The level curves of the function f are $4xy=k$, where k can take different numbers. Later we shall find the right k that gives the maximum area by a method called "method of Lagrange multipliers"!

(2) Lagrange's Theorem (13.19)
Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0,y_0) on the smooth constraint curve $g(x,y)=c$. If $\nabla g(x_0,y_0)\neq\vec{0}$, then there is a real number λ such that

$$\nabla f(x_0,y_0)=\lambda\nabla g(x_0,y_0).$$

(3) Method of Lagrange Multipliers (one constraint)
How to use the Lagrange's theorem?

Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0) on the smooth constraint curve $g(x, y) = c$. The following steps help us find the min or max value of f.

- Step 1: Solve the following system of equations:
  $f_x(x, y) = \lambda g_x(x, y)$
  $f_y(x, y) = \lambda g_y(x, y)$
  $g(x, y) = c$
- Step 2: Evaluate at each solution. The largest is max, and the smallest is the min.
- Note: step one above can be extended to three variables case:
  $f_x(x, y, z) = \lambda g_x(x, y, z)$
  $f_y(x, y, z) = \lambda g_y(x, y, z)$
  $f_z(x, y, z) = \lambda g_z(x, y, z)$
  $g(x, y, z) = c$

(4) Optimization problems with one constraint
- Example 1: Lagrange multiplier and two variables (p. 972)
- Example 2: a business application (p. 973)
- Example 3: Lagrange multiplier and three variables (p. 974)
- Example 4: optimization inside a region (p. 974)

13.10.2. Lagrange multipliers with two constraints

(1) Method of Lagrange Multipliers (two constraints)
With the same setting as in one variable case, except for two constraints, g and h, then $\nabla f = \lambda \nabla g + \mu \nabla h$ will lead to potential extrema.
- Step 1: Solve the following system of equations:
  $f_x(x, y, z) = \lambda g_x(x, y, z) + \mu h_x(x, y, z)$
  $f_y(x, y, z) = \lambda g_y(x, y, z) + \mu h_y(x, y, z)$
  $f_z(x, y, z) = \lambda g_z(x, y, z) + \mu h_z(x, y, z)$
  $g(x, y, z) = c_1$
  $h(x, y, z) = c_2$
- Step 2: Evaluate at each solution. The largest is max, and the smallest is the min.

(2) Optimization problems with two constraints
- Example 5: optimization with two constraints (p. 975)

13.10.3. Homework Set #22

- Read 13.10 (pages 970-975).
- Do exercises on pages 976-977:
  1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 27, 51, 53