Lecture 20

13.8. Extrema of Functions of Two Variables

Goals: (1) Find absolute and relative extrema of a function of two variables.

(2) Use the Second Partials Test to find relative extrema of a function of two variables.

Questions:

- What are relative extrema, and absolute extrema of a function $y = f(x)$?
- How to find relative extrema of a function $y = f(x)$?

13.8.1. **Relative extrema**

Use the gradient and second-order partial derivatives to study the relative extrema (maximum and/or minimum) of a function of two variables.

(1) Definition of Relative extrema:

Let f be a function defined on a region R that contains (x_0, y_0). Then

- (x_0, y_0) is called a ***relative maximum point*** if $f(x,y) \geq f(x_0, y_0)$ for all (x,y) in an open disk containing (x_0, y_0).
- (x_0, y_0) is called a ***relative minimum point*** if $f(x,y) \leq f(x_0, y_0)$ for all (x,y) in an open disk containing (x_0, y_0).
- (x_0, y_0) is called a ***critical point*** if $\nabla f(x_0, y_0) = \vec{0}$ (that is, both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$) or at least one of them does not exist.

(2) When do relative extrema occur?

If f has a relative extremum at (x_0, y_0) on an open region R, then (x_0, y_0) must be a critical point of f.

(3) Examples 1, 2: finding relative extrema (p. 956)

- Try exercises 7-16

13.8.2. **The second partials test**

(1) Let f have continuous second partial derivatives on an open region containing a point (a, b) such that $\nabla f(a,b) = \vec{0}$ (critical point). Let

$$d = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b) f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2.$$

- If $d > 0$ and $f_{xx}(a,b) > 0$, then f has relative minimum at (a, b)
- If $d > 0$ and $f_{xx}(a,b) < 0$, then f has relative maximum at (a, b)
- If $d < 0$, then $\left(a, b, f(a,b)\right)$ is a saddle point.
- If $d = 0$, then the test fails (it is inconclusive).

Notice that $f_{xy}(a,b) = f_{yx}(a,b)$ because they are continuous by assumption.

(2) Examples 3, 4: using the second partials test (p. 958)
- Try exercises 21-28, 31-34

13.8.3. **Absolute extrema**

(1) Definition of Absolute extrema:
Let f be a continuous function defined on a closed region R that contains (x_0, y_0). Then
- (x_0, y_0) is called an ***absolute maximum point*** if $f(x,y) \geq f(x_0, y_0)$ for all (x,y) in R.
- (x_0, y_0) is called an ***absolute minimum point*** if $f(x,y) \leq f(x_0, y_0)$ for all (x,y) in R.

(2) When do absolute extrema occur?
If $f(x,y)$ is a continuous function on a closed bounded region R, then the function has at least one absolute *minimum* point, and at least one absolute *maximum* point. A region is ***bounded*** if it is a subregion of a closed disk in the plane.

(3) Absolute extrema of a function can only occur in two ways:
- Some relative extrema are absolute extrema.
- Some boundary points of the domain are absolute extrema.

(4) Example 5 (p. 959)
- Try exercises 45-54

13.8.4. **Homework Set #20**

- Read 13.8 (pages 954-959).
- Do exercises on pages 960-961:
  7, 9, 11, 13, 15, 21, 23, 25, 27, 31, 33, 45, 49, 51, 53, 61-64