# Lecture 19

## 13.7. Tangent Planes and Normal Lines

**Goals:** (1) Find equations of tangent planes and normal lines to surfaces.

(2) Find the angle of inclination of a plane in space.

(3) Compare the gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$.

Questions:

- Why do we study the gradient of a function $z = f(x, y)$?

13.7.1. **Review: level surface**

Example 1: describe level surface given by $F(x, y, z) = 0$ (p. 945)

- Try exercises 1-4

13.7.2. **Tangent plane and normal line to a surface**

Use an implicit function to study a surface.

(1) Definition:

Let $F$ be a differentiable function at the point $P(x_0, y_0, z_0)$ on the surface $S$ given by $F(x, y, z) = 0$ such that $\nabla F(x_0, y_0, z_0) \neq \vec{0}$. Then

- The plane through $P(x_0, y_0, z_0)$ that is normal to $\nabla F(x_0, y_0, z_0)$ is called the ***tangent plane*** to $S$ at $P$.
- The line through $P(x_0, y_0, z_0)$ that has the direction of $\nabla F(x_0, y_0, z_0)$ is called the ***normal line*** to $S$ at $P$.

(2) How to find an equation of tangent plane to the surface $F(x, y, z) = 0$?

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

(3) How to find an equation of tangent plane to the surface $z = f(x, y)$?

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

Note: Let $F(x, y, z) = f(x, y) - z = 0$.

(4) Examples 2, 3: finding an equation of a tangent plane (pp. 947-948)

- Try exercises 17-30

(5) How to find an equation of normal line to the surface $F(x, y, z) = 0$?

Answer: Use $\nabla F(x_0, y_0, z_0)$ as the directional vector! In other words, the gradient is normal to the surface.

(6) Example 4: finding an equation of a normal line (p. 948)

- Try exercises 31-40

(7) Example 5: finding the equation of a tangent line to a curve (p. 949)

- The cross product of two gradients will be the directional vector of the common tangent line to those two surfaces. See Figure 13.61.
- Try exercises 41-46

### 13.7.3. **The angle of inclination of a plane**

Another use of the gradient is to determine the ***angle of inclination*** of a plane (between the plane and the $xy$-plane, $0 \le \theta \le \frac{\pi}{2}$) that is tangent to a surface.

(1) Let $\vec{n}$ be a normal vector of a plane, and $\vec{k} = \langle 0, 0, 1 \rangle$. Then the angle of inclination of a plane is given by:

$$\cos\theta = \frac{|\vec{n} \cdot \vec{k}|}{\|\vec{n}\|}$$

(2) If $F(x, y, z) = 0$, then
$\nabla F(x, y, z) = F_x(x, y, z)\,\vec{i} + F_y(x, y, z)\,\vec{j} + F_z(x, y, z)\,\vec{k}$.
In particular, if $z = f(x, y)$, then $F(x, y, z) = f(x, y) - z = 0$, and
$\nabla F(x, y, z) = f_x(x, y)\,\vec{i} + f_y(x, y)\,\vec{j} - \vec{k}$.

(3) Example 6: finding the angle of inclination of a tangent plane (p. 950)

- Use the gradient as the <u>normal vector</u> to the tangent plane!
- Try exercises 47-50

### 13.7.4. **The gradient of $F(x, y, z) = 0$**

(1) Recall that in Section 13.6, if $f$ is differentiable at $(x_0, y_0)$ and $\nabla f(x_0, y_0) \neq \vec{0}$, then $\nabla f(x_0, y_0)$ is normal to the <u>level curve</u> through $(x_0, y_0)$.

(2) Similarly, if $F$ is differentiable at $(x_0, y_0, z_0)$ and $\nabla F(x_0, y_0, z_0) \neq \vec{0}$, then the gradient $\nabla F(x_0, y_0, z_0)$ is normal to the <u>level surface</u> through $(x_0, y_0, z_0)$.

(3) Notice that $\nabla f(x, y)$ is a vector in the $xy$-plane and $\nabla F(x, y, z)$ is a vector in space.

### 13.7.5. **Homework Set #19**

- Read 13.7 (pages 945-950).
- Do exercises on pages 951-953:
  2, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 27, 29, 31, 33, 35, 41, 43, 45, 49, 51, 53