Lecture 18

13.6. Directional Derivatives and Gradients

Goals: (1) Find and use directional derivatives of a function of two variables.

(2) Find the gradient of a function of two variables.

(3) Use the gradient of a function of two variables in applications.

(4) Find directional derivatives and gradients of functions of three variables.

Questions:

- What are the partial derivatives of a function $z = f(x, y)$?

13.6.1. **Directional derivative of a function of two variables**

Use the unit vector to study all directions of line paths.

(1) Definition:

Let f be a function of two variables x, y and let $\vec{u} = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath}$ be a unit vector. Then the ***directional derivative*** of f in the direction of $\vec{u}$, denoted by $D_{\vec{u}}f$ is

$$D_{\vec{u}}f(x,y) = \lim_{t\to 0}\frac{f(x + t\cos\theta, y + t\sin\theta) - f(x,y)}{t}$$

provided this limit exists.

(2) Formula 1:

$$D_{\vec{u}}f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

<u>Note</u>: There are infinitely many directional derivatives at a certain point. In particular, when $\theta = 0, \vec{u} = \vec{\imath}, D_{\vec{\imath}}f = f_x$; when $\theta = \frac{\pi}{2}, \vec{u} = \vec{\jmath}, D_{\vec{\jmath}}f = f_y$.

(3) Examples 1, 2: finding a directional derivative (p. 935)

- Try exercises 1-8, 13-16, 17-18

13.6.2. **The gradient of a function of two variables**

The gradient of a function is a vector that can be used to find the directional derivative of the function.

(1) Let $z = f(x, y)$ be a function whose partial derivatives exist. Then the ***gradient*** of f, denoted by $\nabla f(x,y)$, or grad $f(x,y)$, is the vector-valued function:

$$\nabla f(x,y) = f_x(x,y)\,\vec{\imath} + f_y(x,y)\,\vec{\jmath}$$

(2) Formula 2:

$$D_{\vec{u}}f(x,y) = \nabla f(x,y)\cdot\vec{u}$$

(3) Example 3: finding the gradient of a function (p. 936)

- Try exercises 21-24

(4) Example 4: using $\nabla f(x,y)$ to find a directional derivative (p. 937)

- Try exercises 27-30

13.6.3. **Properties of the gradient**

An advanced topic: Operator is a mapping from a set to another set. Here are three examples:
∇ is an "operator" which maps *scalar-valued* functions to *vector-valued* functions. One can consider a regular function to be a special "operator" which maps the *domain* (numbers) to the *range* (still numbers). Another example of "operator" is $\frac{d}{dx}$, which maps *functions* to *functions*. That is, when you apply $\frac{d}{dx}$ to a function, you get another function!

(1) Let f be differentiable at the point (x, y).
- If $\nabla f(x, y) = \vec{0}$, then $D_{\vec{u}} f(x, y) = 0$ for all $\vec{u}$.
- If $\nabla f(x, y) \neq \vec{0}$, $\nabla f(x, y)$ gives the direction of *maximum* increase of f. Furthermore, the *maximum* value of $D_{\vec{u}} f(x, y)$ is $\|\nabla f(x, y)\|$. In other words, f increases most rapidly in the direction of the gradient $\nabla f(x, y)$. See Figure 13.49 (p. 937). Notice that this is a local solution, meaning that different point will lead to different maximum increase.
- Similarly, $-\nabla f(x, y)$ gives the direction of *minimum* increase of f. Furthermore, the *minimum* value of $D_{\vec{u}} f(x, y)$ is $-\|\nabla f(x, y)\|$.

(2) An observation of gradient:
If f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \vec{0}$, then $\nabla f(x_0, y_0)$ is *normal* to the level curve through (x_0, y_0). See Example 6 and Figure 13.52.

(3) Example 5: finding the direction of max increase (p. 938).
- Try exercises 31-36

(4) Example 6: finding the path of a heat-seeking particle (p. 939).
- Try exercises 55-58

(5) Example 7: finding a normal vector $\nabla f(x, y)$ to a level curve (p. 940).
- Try exercises 59-62

13.6.4. **Directional derivative of a function of three variables**

Notice that we are no longer able to visualize the directional derivative for a function in three variables.

(1) Definition:
Let f be a function of three variables x, y, z and let $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}$ be a unit vector. Then the ***directional derivative*** of f in the direction of $\vec{u}$, denoted by $D_{\vec{u}} f$ is

$$D_{\vec{u}} f(x, y, z) = \lim_{t \to 0} \frac{f(x + ta, y + tb, z + tc) - f(x, y, z)}{t}$$

provided this limit exists.

(2) Formula 3:

$$D_{\vec{u}} f(x, y, z) = a f_x(x, y, z) + b f_y(x, y, z) + c f_z(x, y, z)$$

Note: Here we have $a^2 + b^2 + c^2 = 1$, as $\vec{u}$ is a unit vector.
Try exercises 9-12, 19-20

13.6.5. **The gradient of a function of three variables and properties**

This is a natural extension of gradient of a function of two variables.

(1) The gradient is:

$$\nabla f(x,y,z) = f_x(x,y,z)\,\vec{i} + f_y(x,y,z)\,\vec{j} + f_z(x,y,z)\,\vec{k}$$

(2) Formula 4:

$$D_{\vec{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \vec{u}$$

(3) Let f be differentiable at the point (x,y,z).

- If $\nabla f(x,y,z) = \vec{0}$, then $D_{\vec{u}} f(x,y,z) = 0$ for all $\vec{u}$.
- If $\nabla f(x,y,z) \neq \vec{0}$, $\nabla f(x,y,z)$ gives the direction of *maximum* increase of f. Furthermore, the *maximum* value of $D_{\vec{u}} f(x,y,z)$ is $\|\nabla f(x,y,z)\|$. In other words, f increases most rapidly in the direction of the gradient $\nabla f(x,y,z)$. Notice that this is a local solution, meaning that different point will lead to different maximum increase.
- Similarly, $-\nabla f(x,y,z)$ gives the direction of *minimum* increase of f. Furthermore, the *minimum* value of $D_{\vec{u}} f(x,y,z)$ is $-\|\nabla f(x,y,z)\|$.

(4) Example 8: finding the gradient (p. 941)

- Try exercises 25-26, 37-40

13.6.6. **Homework Set #18**

- Read 13.6 (pages 933-941).
- Do exercises on pages 942-944:
  5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 35, 37, 39, 53, 55, 57, 65, 73-76