Lecture 17

13.5. Chain Rules for Functions of Several Variables

Goals: (1) Use the Chain Rules for functions of several variables.

(2) Find partial derivatives implicitly.

Questions:

- What is the chain rule for a function $y = f(x)$?
- What is the implicit differentiation for $y = f(x)$?

13.5.1. **Chain rules with one variable for functions of several variables**

The chain rules provide alternative techniques for solving many problems in single-variable calculus.

(1) Let $w = f(x, y)$, $x = g(t)$, and $y = h(t)$ be differentiable functions. Then $w = f\big(g(t), h(t)\big)$ is a differentiable function in one variable t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

Note: This formula can be extended to 3 or more variables.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

(2) Example 1: using the chain rule with one independent variable (p. 925)
- Try exercises 1-4, 5-10

(3) Example 2: application of chain rule to related rates (p. 926)
- Try exercises 11-12

13.5.2. **Chain rule with two independent variables**

(1) Let $w = f(x, y)$, $x = g(s, t)$, and $y = h(s, t)$ be differentiable functions and all of the partial derivatives exist. Then $w = f\big(g(s, t), h(s, t)\big)$ is a differentiable function in two variables s, t, and

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}, \text{ and } \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$

Note: This formula can be extended to 3 or more variables.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}, \text{ and } \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}.$$

(2) Example 3: finding partial derivatives by substitution (p. 927)

(3) Example 4: finding partial derivatives using the chain rules (p. 928)
- Try exercises 15-18, 19-22

(4) Example 5: finding partial derivatives using the chain rules for three variables (p. 929)
- Try exercises 23-26

13.5.3. **Implicit partial differentiation**

(1) If $F(x,y)=0$ is an implicit function, then the derivative $\frac{dy}{dx}$ can be found implicitly:

$$\frac{dy}{dx}=-\frac{F_x(x,y)}{F_y(x,y)}, \text{where } F_y(x,y)\neq 0.$$

Proof: Let $w=F(x,y)=0$. Use the chain rule $0=\frac{dw}{dx}=\frac{\partial w}{\partial x}\frac{dx}{dx}+\frac{\partial w}{\partial y}\frac{dy}{dx}$.

(2) If $F(x,y,z)=0$ is an implicit function, then the two partial derivatives can be found implicitly:

$$\frac{\partial z}{\partial x}=-\frac{F_x(x,y,z)}{F_z(x,y,z)} \text{ and } \frac{\partial z}{\partial y}=-\frac{F_y(x,y,z)}{F_z(x,y,z)}, \text{where } F_z(x,y,z)\neq 0.$$

Proof: Let $w=F(x,y,z)=0$. Use the chain rules: $0=\frac{\partial w}{\partial x}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial x}$ and $0=\frac{\partial w}{\partial y}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial y}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial y}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$. Notice that $\frac{\partial y}{\partial x}=0=\frac{\partial x}{\partial y}$ as x and y are independent variables.

Note: This can be extended to $F(x,y,z,w)=0$. See Exercises 39-42.

(3) Example 6: finding derivative implicitly (p. 930).
- Try exercises 27-30

(4) Example 7: finding partial derivatives implicitly (p. 930).
- Try exercises 31-38

13.5.4. **Homework Set #17**

- Read 13.5 (pages 925-930).
- Do exercises on pages 931-932:
  1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 35, 37, 39, 53, 55, 59, 61