Lecture 16

13.4. Differentials

Goals: (1) Understand the concepts of increments and differentials.

(2) Extend the concept of differentiability to a function of two variables.

(3) Use the differential as an approximation.

Questions:

- Q: What is the definition of the differential of a function $y = f(x)$?
  A: $dy = f(x)dx$

13.4.1. **Increments and differentials**

(1) Definition: The ***increment*** of $z = f(x, y)$ is given by:

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

where $\Delta x, \Delta y$ are increments of x and y, respectively.

(2) Definition: The ***differentials*** of the independent variables x and y are $dx = \Delta x$ and $dy = \Delta y$. The ***total differential*** of the dependent variable z is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f_x(x, y)dx + f_y(x, y)dy$$

(3) Example 1: finding the total differential (p. 918)

- Exercises 1-10

13.4.2. **Differentiability**

(1) $z = f(x, y)$ is ***differentiable*** at (x_0, y_0) if

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where both ε_1 and ε_2 approach 0 as $(\Delta x, \Delta y) \to (0, 0)$.
The function is ***differentiable in a region*** R if it is differentiable at each point in R.

(2) If f is a function of x and y, where f_x and f_y are continuous in an open region R, then f is differentiable on R.

(3) Differentiability implies continuity, but not the other way round.
Therefore, non-Continuous implies non-Differentiable.
See Example 5 (p. 922)

(4) Example 2: showing that a function is differentiable (p. 919).

13.4.3. **Approximation by differentials**

(1) Recall: $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$ and $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$.
$\Delta z \approx dz$ is called a ***linear approximation***.

(2) Example 3: using differential as an approximation (p. 920).

- Try exercises 11-16, 17-20

13.4.4. **Differentials of functions in three variables**

(1) Definition: The ***increment*** of $w = f(x, y, z)$ is given by:

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

where $\Delta x, \Delta y, \Delta z$ are increments of x y, and z, respectively.

(2) $w = f(x, y, z)$ is ***differentiable*** at (x, y, z) if

$$\Delta w = f_x(x, y, z)\Delta x + f_y(x, y, z)\Delta y + f_z(x, y, z)\Delta z + \varepsilon_1\Delta x + \varepsilon_2\Delta y + \varepsilon_3\Delta z$$

where ε_1, ε_2 and ε_3 approach 0 as $(\Delta x, \Delta y, \Delta z) \to (0, 0, 0)$.

(3) If f is a function of x, y, and z, where f, f_x, f_y, and f_z are continuous in an open region R, then f is differentiable on R.

(4) Example 4: error analysis
Try exercises 25-40

13.4.5. **Applications**

Exercise 36.

(a) Using the law of cosines: $a \approx 107.3$ ft.

(b) $da \approx \pm 8.27$ ft.

13.4.6. **Homework Set #16**

- Read 13.4 (pages 918-922).
- Do exercises on pages 923-924:
  1, 3, 5, 7, 9, 11, 13, 17, 19, 25, 27, 35, 37