Lecture 15

13.3. Partial Derivatives

Goals: (1) Find and use partial derivatives of a function of two variables.
(2) Find and use partial derivatives of a function of three or more variables.
(3) Find higher-order partial derivatives of a function of two or three variables.

Questions:

- What is the definition of the derivative of a function?
- How to find the derivative of a function $y = f(x)$?

13.3.1. **Partial derivatives of a function of two variables**

(1) Definition: The ***first partial derivatives*** of $z = f(x, y)$ with respect to x and y are the functions f_x and f_y defined by the following limits:

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$
$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

provided that the limits exist.

(2) Notations for first partial derivatives
- $f_x(x, y) = z_x = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x}$
- $f_y(x, y) = z_y = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y}$
- $f_x(a, b) = \left.\frac{\partial z}{\partial x}\right|_{(a,b)}$
- $f_y(a, b) = \left.\frac{\partial z}{\partial y}\right|_{(a,b)}$

(3) Geometric interpretation:
- $\frac{\partial}{\partial x} f(a, b)$ represents the slope of the curve: $y = b$, $z = f(x, y)$ at point $\big(a, b, f(a, b)\big)$. That is, slope of surface in x-direction.
- $\frac{\partial}{\partial y} f(a, b)$ represents the slope of the curve: $x = a$, $z = f(x, y)$ at point $\big(a, b, f(a, b)\big)$. That is, slope of surface in y-direction.

(4) Examples 1, 2: finding partial derivatives (pp. 908-909)
- Exercises 9-40, 45-52

(5) Examples 3, 4: finding slopes of a surface in both directions (p. 910)
- Exercises 53-54

(6) Example 5: finding rates of change (p. 911)
- Exercises

13.3.2. **Partial derivatives of a function of three or more variables**

The concept of partial derivatives can be extended naturally to functions of three or more variables.

(1) Let $w = f(x, y, z)$ be a function in three variables. Then there will be three partial derivatives:

$$\frac{\partial w}{\partial x} = f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$
$$\frac{\partial w}{\partial y} = f_y(x, y, z) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$
$$\frac{\partial w}{\partial z} = f_z(x, y, z) = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

(2) In general, let $w = f(x_1, x_2, \dots, x_n)$. There are n partial derivatives:

$$\frac{\partial w}{\partial x_k} = f_{x_k}(x_1, x_2, \dots, x_n), k = 1, 2, \dots, n.$$

Note: to find the partial derivative with respect to one of the variables, hold the other variables constant and differentiate with respect to the given variable.

(3) Example 6: finding partial derivatives (p. 912).
- Try exercises 59-64, 65-70

13.3.3. **Higher-order partial derivatives**

(1) 2nd-order partial derivatives:
- $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$
- $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$
- $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$
- $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$

Note: the middle two cases are called mixed partial derivatives.

(2) Example 7: finding second partial derivatives (p. 913).
- Try exercises 71-80

(3) Example 8: finding higher-order partial derivatives of a function in three variables (p. 913).
- Try exercises 93-96

13.3.4. **Homework Set #15**

- Read 13.3 (pages 908-913).
- Do exercises on pages 914-917:
  9, 11, 13, 17, 19, 21, 23, 25, 27, 33, 35, 45, 51, 53, 59, 61, 65, 73, 75, 79, 81, 83, 93, 99, 129-132