# Lecture 14

## 13.2. Limits and Continuity

**Goals:** (1) Understand the definition of a neighborhood in the plane.

(2) Understand and use the definition of the limit of a function of two variables.

(3) Extend the concept of continuity to a function of two variables.

(4) Extend the concept of continuity to a function of three variables.

Questions:

- What is open interval, or closed interval on a number line?
- What is the definition of limit of a function $y = f(x)$? How to determine whether the limit exists or not?
- What is the definition of continuity of a function $y = f(x)$?

13.2.1. **Neighborhoods in the plane**

(1) Open disc: ***δ-neighborhood*** about $(x_0,\ y_0)$

$$\left\{(x, y): \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta\right\}$$

Closed disc:

$$\left\{(x, y): \sqrt{(x-x_0)^2 + (y-y_0)^2} \le \delta\right\}$$

(2) Interior point & Boundary point
A point $(x_0, y_0)$ in a plane region $R$ is an ***interior point*** of $R$ if there exists a $\delta$-neighborhood about $(x_0, y_0)$ that lies entirely in $R$. A point in $R$ is called a ***boundary point*** if any $\delta$-neighborhood about the point contains points inside $R$ and points outside $R$. (See Figure 13.19)

(3) Open region
If every point in $R$ is interior point, then $R$ is called an ***open region***.

(4) Closed region
If the region contains all its boundary points, then $R$ is called a ***closed region***.

13.2.2. **Limit of a function of two variables**

(1) Let $z = f(x, y)$ be a function defined on an open disk of radius $\delta$ (excluding the center). Then the limit:

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if for each $\varepsilon > 0$, there is a $\delta > 0$ such that
whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta, 0 < |f(x, y) - L| < \varepsilon$.

(2) How to determine whether the limit exists?

- All paths/directions of $(x, y)$ approaching $(x_0, y_0)$ should lead to the same limit.
- Properties of limit:
  Scalar multiple, Sum, Difference, Product, Quotient.

(3) Example 1: verifying a limit by definition (p. 899).
- Try exercises 1-4

(4) Example 2: verifying a limit by substitution (p. 900).
(5) Example 3: verifying a limit by squeezing theorem (p. 900).
(6) Example 4: showing nonexistence of a limit (p. 901).
- Try exercises 5-8, 19-24

13.2.3. **Continuity of a function of two variables**

(1) A function $z = f(x, y)$ is said to be ***continuous at*** $(x_0, y_0)$ in an open region $R$ if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

If the function is continuous at every point in the open region $R$, then we say that the function is ***continuous in the open region*** $R$.

(2) Removable and nonremovable discontinuity:
- If $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists, but $\neq f(x_0, y_0)$, or $f(x_0, y_0)$ is not defined, then it is removable in both cases.
- If $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist, then it is not removable.

(3) Properties of continuity:
Scalar multiple, Sum, Difference, Product, Quotient, Composite. (pp. 902)

(4) Example 5: testing for continuity (p. 903).
- Try exercises 9-18, 25-28, 33-34, 41-48, 55-58

13.2.4. **Continuity of a function of three variables**

Open sphere: $\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta$

(1) A function $z = f(x, y, z)$ is said to be ***continuous at*** $(x_0, y_0, z_0)$ in an open region $R$ if

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = f(x_0, y_0, z_0)$$

If the function is continuous at every point in the open region $R$, then we say that the function is ***continuous in the open region*** $R$.

(2) Example 6: testing for continuity (p. 904).
- Try exercises 49-54, 73-74

13.2.5. **Homework Set #14**

- Read 13.2 (pages 898-904).
- Do exercises on pages 904-907:
  7, 8, 11, 13, 17, 19, 21, 23, 25, 27, 29, 31, 35, 43, 45, 53, 55, 57, 59, 61, 67, 71, 73, 79-82, 85