Lecture 12

12.5. Arc Length and Curvature

Goals: (1) Find the arc length of a space curve.

(2) Use the arc length parameter to describe a plane curve or space curve.

(3) Find the curvature of a curve at a point on the curve.

(4) Use a vector-valued function to find frictional force.

Questions:

- What is the formula for the length of a smooth plane curve C given by the parametric equations $x = x(t),\ y = y(t)$on an interval $[a, b]$?
  Answer: $s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$.
- What is the formula, if the curve is given in vector form, that is,
  $$\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath}?$$
  Answer: $s = \int_a^b \|\vec{r}'(t)\|\, dt$.
  Try exercises 1-6 (p. 877)
- Recall:
  $\vec{T}(s) = \dfrac{\vec{r}'(s)}{\|\vec{r}'(s)\|}$ is the unit tangent vector of $\vec{r}(s)$.
  $\vec{a}(t) = a_{\vec{T}}\vec{T}(t) + a_{\vec{N}}\vec{N}(t)$ is the decomposition of acceleration.

12.5.1. **Arc length of a space curve**

(1) If C is a smooth curve (in space) given by $\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$, on an interval $[a, b]$, then the arc length of C on the interval is:

$$s = \int_a^b \|\vec{r}'(t)\|\, dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$

(2) Examples 1, 2: finding the arc length of a curve in space (pp. 869-870).

- Try exercises 9-14

12.5.2. **Arc length parameter**

Curves can be represented by vector-valued functions in different way, depending upon the choice of parameter. For *motion* along a curve, the convenient parameter is time t, while for studying *geometric properties* of a curve, the convenient parameter is arc length s.

(1) Definition: Let C be a smooth curve given by $\vec{r}(t)$ on a closed interval $[a, b]$. For $a \le t \le b$, the ***arc length function*** is given by

$$s(t) = \int_a^t \|\vec{r}'(w)\|\, dw = \int_a^t \sqrt{[x'(w)]^2 + [y'(w)]^2 + [z'(w)]^2}\, dw$$

Note: $s(t) \geq 0$. It's also called arc length parameter. It measures the distance along C from the initial point $(x(a), y(a), z(a))$ to $(x(t), y(t), z(t))$.

(2) By the Second Fundamental Theorem of Calculus,

$$\frac{ds}{dt} = \|\vec{r}'(t)\|, \text{ or}$$
$$ds = \|\vec{r}'(t)\|dt.$$

Note: The second formula is in differential form. Now if we choose s as the parameter, that is, $\vec{r}(s) = x(s)\vec{\imath} + y(s)\vec{\jmath} + z(s)\vec{k}$, then $\|\vec{r}'(s)\| = 1$. On the other hand, if $\|\vec{r}'(t)\| = \frac{ds}{dt} = 1$, then t must be the arc length parameter. Therefore, the parameter t is the arc length parameter if and only if $\|\vec{r}'(t)\| = 1$.

(3) Example 3: finding the arc length function for a line (p. 871).
Hint: solve for t in terms of s.

- Try exercises 19, 20

12.5.3. **Curvature**

Another important use of the arc length parameter is to find curvature—the measure of how sharply a curve bends.

(1) Definition: Let C be a smooth curve given by $\vec{r}(s)$ where s is the arc length parameter. The curvature K at s is given by

$$K = \left\|\frac{d\vec{T}}{ds}\right\| = \|\vec{T}'(s)\|.$$

(2) In general, if the parameter is t (not necessary to be s), then the curvature at t is given by

$$K = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}.$$

(3) Example 5: finding the curvature of a space curve (p. 873).

- Try exercises 21-40

(4) In particular, if C is given by a regular (rectangular coordinates) function $y = f(x)$, then the curvature at $P(x, y)$ is given by

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}}$$

- Proof: $\vec{r}(t) = x\vec{\imath} + f(x)\vec{\jmath} + 0\vec{k}$, $\vec{r}'(t) = \vec{\imath} + f'(x)\vec{\jmath}$, and $\vec{r}''(t) = f''(x)\vec{\jmath}$. So $\vec{r}'(t) \times \vec{r}''(t) = f''(x)\vec{k}$.
- Definition: The circle (on the concave side of the curve) with center P and radius $r = 1/K$ is called the ***circle of curvature*** (See Figure 12.36 or 12.37). P is called the *center* of curvature, r is called the *radius* of curvature at P.

(5) Example 6: finding the curvature of a plane curve (p. 874).

- Try exercises 41-44, 45-54

(6) Relationship among acceleration, speed, and curvature.

$$\vec{a}(t) = \frac{d^2s}{dt^2}\vec{T} + K\left(\frac{ds}{dt}\right)^2 \vec{N}$$

(7) Example 7: finding the tangential and normal components of acceleration (p. 875).

- Try exercise 96

(8) Application in physics and engineering dynamics: <u>frictional force</u>

$$\vec{F} = m\vec{a} = m\frac{d^2s}{dt^2}\vec{T} + mK\left(\frac{ds}{dt}\right)^2\vec{N}$$

$mK\left(\frac{ds}{dt}\right)^2$ is called the ***force of friction***.

(9) Example 8: finding frictional force (p. 876).

- Try exercises 97-98

(10) Summary: p. 877. <u>This is a very important list of formulas</u>.

12.5.4. **Homework Set #12**

- Read 12.5 (pages 869-877).
- Do exercises on pages 877-880:
  1, 3, 5, 7, 9, 11, 13, 21, 23, 27, 29, 31, 33, 35, 37, 41, 43, 87, 97, 101-104