Lecture 11

12.4. Tangent Vectors and Normal Vectors

Goals: (1) Find a unit tangent vector at a point on space curve.

(2) Find the tangential and normal components of acceleration.

Questions:

- What is the definition of smooth curve $\vec{r}$ on an interval?
  Answer: $\vec{r}'$ is continuous and nonzero on the interval.
- Is $\vec{r}'$ orthogonal to $\vec{r}$?
  Answer: see page 844, last line.

12.4.1. **Tangent vectors**

(1) Definition: Let C be a smooth curve represented by $\vec{r}$ on an open interval. The ***unit tangent vector*** at t is defined to be

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}, \vec{r}'(t) \neq \vec{0}.$$

Note: $\vec{T}(t) \cdot \vec{T}(t) = 1$.

(2) Example 1: finding the unit tangent vector (p. 859).
- Try exercises 5-10

(3) Example 2: finding the tangent line (p. 860).
- Try exercises 11-16

12.4.2. **Normal vectors**

(1) Definition: Let C be a smooth curve represented by $\vec{r}$ on an open interval. The ***principal unit normal vector*** at t is defined to be

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}, \vec{T}'(t) \neq \vec{0}.$$

Note: $\vec{T}(t)$ and $\vec{N}(t)$ are orthogonal to each other, i.e., $\vec{T}(t) \cdot \vec{N}(t) = 0$.

(2) For plane curves, there is an easy way to find principal unit normal vector. Here are the steps:
- Find $\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = x(t)\vec{\imath} + y(t)\vec{\jmath}$.
- Then we can take $\vec{N}(t) = y(t)\vec{\imath} - x(t)\vec{\jmath}$, or $\vec{N}(t) = -y(t)\vec{\imath} + x(t)\vec{\jmath}$.

(3) Examples 3, 4: finding the unit tangent vector and principal unit normal vector (pp. 861-862).
- Try exercises 23-30, 31-34

12.4.3. **Acceleration vectors**

(1) Recall: If $\vec{r}$ is the position vector for a smooth curve C. Then $\vec{v} = \vec{r}'$ and $\vec{a} = \vec{v}' = \vec{r}''$. Notice that:

$$\vec{v} = \vec{r}' = \|\vec{r}'\|\vec{T} = \|\vec{v}\|\vec{T}$$
$$\vec{a} = \vec{v}' = \|\vec{v}\|'\vec{T} + \|\vec{v}\|\vec{T}' = \|\vec{v}\|'\vec{T} + \|\vec{v}\|\|\vec{T}'\|\vec{N}$$

(2) If $\vec{N}(t)$ exists (which implies that $\vec{T}(t)$ also exists), then

$$\vec{a}(t) = a_{\vec{T}}\vec{T}(t) + a_{\vec{N}}\vec{N}(t)$$

where

$$a_{\vec{T}} = \|\vec{v}\|' = \vec{a}\cdot\vec{T} = \frac{\vec{v}\cdot\vec{a}}{\|\vec{v}\|}$$
$$a_{\vec{N}} = \|\vec{v}\|\|\vec{T}'\| = \vec{a}\cdot\vec{N} = \frac{\|\vec{v}\times\vec{a}\|}{\|\vec{v}\|} = \sqrt{\|\vec{a}\|^2 - {a_{\vec{T}}}^2}$$

Proof 1:

$\vec{v}\times\vec{a} = \|\vec{v}\|\vec{T}\times\left[a_{\vec{T}}\vec{T} + a_{\vec{N}}\vec{N}\right] = \vec{0} + \|\vec{v}\|a_{\vec{N}}\left[\vec{T}\times\vec{N}\right] = \|\vec{v}\|a_{\vec{N}}\left[\vec{T}\times\vec{N}\right]$, where $\vec{T}\times\vec{N}$ is a unit vector orthogonal to $\vec{T},\vec{N}$. So, $\|\vec{v}\times\vec{a}\| = \|\vec{v}\|a_{\vec{N}}$.

Proof 2:

$\|\vec{a}\|^2 = \vec{a}\cdot\vec{a} = \left[a_{\vec{T}}\vec{T}(t) + a_{\vec{N}}\vec{N}(t)\right]\cdot\left[a_{\vec{T}}\vec{T}(t) + a_{\vec{N}}\vec{N}(t)\right] = {a_{\vec{T}}}^2 + {a_{\vec{N}}}^2$. So, ${a_{\vec{N}}}^2 = \|\vec{a}\|^2 - {a_{\vec{T}}}^2$. Notice that $a_{\vec{N}} = \|\vec{v}\|\|\vec{T}'\| \geq 0$.

(3) Examples 5, 6, 7: finding $a_{\vec{T}}$ and $a_{\vec{N}}$ (pp. 864-865).

- Try exercises 35-44, 55-62

12.4.4. **Homework Set #11**

- Read 12.4 (pages 859-865).
- Do exercises on pages 865-868:
  5, 7, 9, 11, 13, 15, 23, 25, 27, 29, 31, 33, 35, 37, 41, 53, 55, 57, 61, 91-92