Lecture 9

12.2. Differentiation and Integration of Vector-Valued Functions

Goals: (1) Differentiate a vector-valued function.

(2) Integrate a vector-valued function.

12.2.1. **Review**

(1) Basic differentiation rules:

- Constant multiplication rule: $\frac{d}{dx}(cf) = c\frac{d}{dx}(f)$, where c is a constant
- Sum/Difference rule: $\frac{d}{dx}(f \pm g) = \frac{d}{dx}(f) \pm \frac{d}{dx}(g)$
- Product rule: $\frac{d}{dx}(f \cdot g) = \frac{d}{dx}(f)g + f\frac{d}{dx}(g)$
- Quotient rule: $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{d}{dx}(f)g - f\frac{d}{dx}(g)}{g^2}$
- Chain rule: $\frac{d}{dx}(f \circ g) = \left(\frac{d}{du}(f)\right)\left(\frac{d}{dx}(u)\right)$
- $\frac{d}{dx}(x^n) = nx^{n-1}$, esp., $\frac{d}{dx}(1) = 0$
- $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$, where $b > 0$, esp., $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- $\frac{d}{dx}(a^x) = a^x \ln a$, where $a > 0$, esp., $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\csc^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\csc x) = -\csc x \cot x$
- $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\operatorname{arccot} x) = -\frac{1}{1+x^2}$
- $\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}(\operatorname{arccsc} x) = -\frac{1}{|x|\sqrt{x^2-1}}$

(2) Basic integration rules:

- $\int cf(t)dt = c\int f(t)dt$, where c is a constant
- $\int [f(t) \pm g(t)]dt = \int f(t)dt \pm \int g(t)dt$
- $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$, where $n \neq -1$, esp., $\int 0dx = C$
- $\int x^{-1} dx = \ln|x| + C$
- $\int a^x dx = \frac{a^x}{\ln a} + C$, esp., $\int e^x dx = e^x + C$
- $\int \cos x\, dx = \sin x + C$
- $\int \sin x\, dx = -\cos x + C$
- $\int \sec^2 x\, dx = \tan x + C$
- $\int \csc^2 x\, dx = -\cot x + C$
- $\int \sec x \tan x\, dx = \sec x + C$
- $\int \csc x \cot x\, dx = -\csc x + C$
- $\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin\frac{x}{a} + C = -\arccos\frac{x}{a} + C$
- $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\arctan\frac{x}{a} + C = -\frac{1}{a}\text{arccot}\frac{x}{a} + C$

12.2.2. **Definition of the derivative of a vector-value function**

$$\vec{r}'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

- Notations: $\vec{r}'(t)$, $D_t[\vec{r}(t)]$, $\frac{d}{dt}[\vec{r}(t)]$, or simply, $\frac{d\vec{r}}{dt}$.

12.2.3. **Differentiation of vector-valued functions**

(1) Differentiation of vector-value based functions can be done on a *component-by-component basis*. That is, let $\vec{r}(t) = f(t)\vec{\imath} + g(t)\vec{\jmath} + h(t)\vec{k}$. Then

$$\vec{r}'(t) = f'(t)\vec{\imath} + g'(t)\vec{\jmath} + h'(t)\vec{k}$$

- Example 1: Differentiation
  Try exercises 11-22
- Example 2: Higher-order differentiation
  Try exercises 23-30

(2) $\vec{r}(t) = f(t)\vec{\imath} + g(t)\vec{\jmath} + h(t)\vec{k}$ is said to be ***smooth***, if f', g', and h' are continuous and not all of them are 0, that is, $\vec{r}'(t) \neq \vec{0}$.

- Example 3: Smooth intervals
  Try exercises 33-42

12.2.4. **Properties of the derivative**

- $D_t[c\vec{r}(t)] = cD_t[\vec{r}(t)]$, where c is a constant
- $D_t[\vec{r}(t) \pm \vec{u}(t)] = D_t[\vec{r}(t)] \pm D_t[\vec{u}(t)]$
- $D_t[f(t)\vec{r}(t)] = f'(t)\vec{r}(t) + f(t)D_t[\vec{r}(t)]$
- $D_t[\vec{r}(t) \cdot \vec{u}(t)] = D_t[\vec{r}(t)] \cdot \vec{u}(t) + \vec{r}(t) \cdot D_t[\vec{u}(t)]$
- $D_t[\vec{r}(t) \times \vec{u}(t)] = D_t[\vec{r}(t)] \times \vec{u}(t) + \vec{r}(t) \times D_t[\vec{u}(t)]$
- $D_t[\vec{r}(f(t))] = \vec{r}'(f(t)) \cdot f'(t)$

- If $\vec{r}(t)\cdot\vec{r}(t)=c$, then $\vec{r}(t)\cdot\vec{r}\,'(t)=0$, where c is a constant

- Example 4: Derivative properties
  Try exercises 43, 44

12.2.5. **Integration of vector-valued functions**

Integration of vector-value based functions can also be done on a *component-by-component basis*. That is, let $\vec{r}(t)=f(t)\vec{\imath}+g(t)\vec{\jmath}+h(t)\vec{k}$. Then

$$\int \vec{r}(t)dt=\left[\int f(t)dt\right]\vec{\imath}+\left[\int g(t)dt\right]\vec{\jmath}+\left[\int h(t)dt\right]\vec{k}$$

$$\int_a^b \vec{r}(t)dt=\left[\int_a^b f(t)dt\right]\vec{\imath}+\left[\int_a^b g(t)dt\right]\vec{\jmath}+\left[\int_a^b h(t)dt\right]\vec{k}$$

- Examples 5, 6, 7: Integration
  Try exercises 49-66

12.2.6. **Homework Set #9**

- Read 12.2 (pages 842-847).
- Do exercises on pages 848-849:
  3, 5, 9, 11, 13, 15, 17, 23, 25, 27, 29, 35, 39, 41, 43, 51, 55, 59, 61, 63, 67, 71, 89-92