Lecture 5

11.5. Lines and Planes in Space

Goals: (1) Write a set of parametric equations for a line in space.

(2) Write a linear equation to represent a plane in space.

(3) Sketch the plane given by linear equation.

(4) Find the distances between points, planes, and lines in space.

Questions:

- What is point-slope formula for a line in plane?
- How to find the angle between two vectors?

11.5.1. Lines in space

(1) The ***point-vector*** equation:

- Given a point $P(x_1, y_1, z_1)$, and the direction vector $\vec{v} = \langle a, b, c\rangle$. Then $\langle x - x_1,\ y - y_1, z - z_1\rangle = t\vec{v}$ represents an equation of the line through point P and parallel to vector $\vec{v}$, where t is any real number.
  Note: different t represents different point on the line.

(2) The ***parametric*** equations:

- Given a point $P(x_1, y_1, z_1)$, and the direction vector $\vec{v} = \langle a, b, c\rangle$. Then the set of equations $x = x_1 + at,\ y = y_1 + bt, z = z_1 + ct$ represents the line through point P and parallel to vector $\vec{v}$, where t is any real number.

(3) The ***symmetric*** equations:

- Given a point $P(x_1, y_1, z_1)$, and the direction vector $\vec{v} = \langle a, b, c\rangle$, where a, b, c are nonzero. Then $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ represents the equations of the line.
  Note: parametric equations or symmetric equations of a line are *not* unique.

(4) Example 1: Finding parametric equations and symmetric equations.

- Try exercises 5-10

(5) Example 2: Finding parametric equations of the line through two points.

- Try exercises 11-14.

11.5.2. Planes in space

An equation of a plane in space can be obtained from a point and a vector normal (perpendicular) to the plane.

(1) ***Standard form*** of the equation:

Given a point $P(x_1, y_1, z_1)$, and a normal vector $\vec{n} = \langle a, b, c\rangle$. Then the plane can be represented by the equation:

$a(x - x_1) + b(\ y - y_1) + c(z - z_1) = 0$.

(2) ***General form*** of the equation:
Given a point $P(x_1, y_1, z_1)$, and a normal vector $\vec{n} = \langle a, b, c \rangle$. Then the plane can be represented by the equation: $ax + by + cz + d = 0$.
Note: the coefficients of the equation are components of the normal vector.
(3) Example 3: Finding equation of a plane with 3 points.
- Try exercises 47-58.

11.5.3. Angles between two planes in space

(1) Let θ be the angle between two intersecting planes. Let $\vec{n}_1$ and $\vec{n}_2$ be the normal vectors of those planes, respectively. Then $\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\|\|\vec{n}_2\|}$.
(2) In particular, two planes are ***perpendicular*** if $\vec{n}_1 \cdot \vec{n}_2 = 0$; two planes are ***parallel*** if $\vec{n}_1$ is a scalar multiple of $\vec{n}_2$.
(3) Example 4: Finding angle between two planes, and the line of intersection of two planes (usually in parametric form).
- Note: The cross product $\vec{n}_1 \times \vec{n}_2$ is parallel to the direction vector of the intersecting line. Why is it true?
- Try exercises 91-92.

11.5.4. Special planes in space

(1) Definition:
If a plane in space intersects one of the coordinate plane, the line of intersection is called the ***trace*** of the given plane in the coordinate plane. See Figure 11.49.
(2) Plane is parallel to an axis if its equation is missing a variable. See Figure 11.50.
(3) Plane is parallel to a coordinate plane if its equation is missing two variables. See Figure 11.51.

11.5.5. Distance between a point and a plane in space

(1) Let $\vec{n}$ be the normal vector of a plane $ax + by + cz + d = 0$ and P be *any* point in the plane. Let $Q(x_0, y_0, z_0)$ be a point not in the plane. Then the distance between Q and the plane is:

$$D = \left\|\text{proj}_{\vec{n}} \overrightarrow{PQ}\right\| = \frac{\left|\overrightarrow{PQ} \cdot \vec{n}\right|}{\|\vec{n}\|}$$

Question: How to find a pint P in the plane?
Answer: Let $y = 0$, and $z = 0$. Then solve for x.
(2) Another nice form:
Given a plane $ax + by + cz + d = 0$ and a point $Q(x_0, y_0, z_0)$.

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Question: What happens if Q is in the plane?
Answer: $D = 0$. Why?
(3) Example 5: Finding the distance between a point and a plane.
- Try exercises 97-100
(4) Example 6: Finding the distance between two parallel planes.

- Try exercises 109-110

11.5.6. Distance between a point and a line in space

(1) Let $\vec{u}$ be the direction vector of a line and P be *any* point on the line. Let Q be a point not on the line. Then the distance between Q and the line is:

$$D = \frac{\left\|\overrightarrow{PQ} \times \vec{u}\right\|}{\|\vec{u}\|}$$

Note: The distance is the shortest length from Q to P on the line. If Q is also on the line, then the distance is zero.

(2) Example 7: Finding the distance between a point and a line.

- Try exercises 105-108

11.5.7. Homework Set #5

- Read 11.5 (pages 798-805).
- Do exercises on pages 805-809:
  5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 39, 41, 43, 45, 47, 49, 53, 55, 57, 59, 63, 65, 67, 71, 73, 83, 85, 91, 95, 97, 99, 101, 103, 105, 107, 109, 129-134