# Lecture 4

# 11.4. The Cross Product of Two Vectors in Space

**Goals:** (1) Find the cross product of two vectors in space.

(2) Use the triple scalar product of three vectors in space.

Questions:

- What is matrix? Square matrix?
- What is the determinant of a matrix?

### 11.4.1. The cross product of two vectors

(1) Definition:

- $\langle u_1, u_2, u_3\rangle \times \langle v_1, v_2, v_3\rangle$, the ***cross product*** of two vectors $\langle u_1, u_2, u_3\rangle$ and $\langle v_1, v_2, v_3\rangle$, is $\langle u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1\rangle$
- Notes:
  - The cross product of two vectors in space… not plane!
  - The cross product is always a vector, not a scalar (number)!
- Another notation:

$$\left(u_1\vec{i} + u_2\vec{j} + u_3\vec{k}\right) \times \left(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

where $\left|\quad\right|$ stands for the determinant of a square matrix.

(2) Example 1: Finding the cross product

- Try exercises 1-10

(3) Some algebraic properties of the cross product

- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
- $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$
- Zero-multiplication: $\vec{0} \times \vec{v} = \vec{0}$. What about $\vec{v} \times \vec{0}$?
- Self-multiplication: $\vec{v} \times \vec{v} = \vec{0}$. Why?
- Triple scalar product: $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

(4) Some geometric properties of the cross product

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in space, and $\theta$ be the angle between $\vec{u}$ and $\vec{v}$.

- $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$. Why?

  Note: What about $\vec{i} \times \vec{j}$ and so on?
- $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$.

  Remark: That's the way to calculate the norm of the cross product.

- $\vec{u} \times \vec{v} = \vec{0}$ if and only if $\vec{u}$ and $\vec{v}$ are parallel ($\theta$ is 0 or $\pi$).
  Remark: That is, $\vec{u}$ is a scalar multiple of $\vec{v}$.
- $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$ is equal to the area of parallelogram having $\vec{u}$ and $\vec{v}$ as adjacent sides. See Figure 11.35.

(5) Examples 2, 3 (page 795)

- Try exercises 11-16, 27-36.

**11.4.2. Torque**

(1) Definition:
If the point of application of the force $\vec{F}$ is $Q$, the moment $\vec{M}$ of force $\vec{F}$ about $P$ is called ***torque***. It can be measured by the formula: $\vec{M} = \overrightarrow{PQ} \times \vec{F}$.

(2) Example 4 (page 796)

- Try exercises 37-38.

**11.4.3. Triple scalar product of three vectors in space**

(1) Definition:
Let $\vec{u}, \vec{v}, \vec{w}$ be three vectors in space. $\vec{u} \cdot (\vec{v} \times \vec{w})$ is called the ***triple scalar product***.

(2) Algebraic property:
$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

(3) Geometric property:
$|\vec{u} \cdot (\vec{v} \times \vec{w})| = V$ is the volume of a parallelepiped with vectors $\vec{u}, \vec{v}, \vec{w}$ as the adjacent edges. See Figure 11.41

(4) Example 5 (page 797):
Finding volume by the triple scalar product.

- Try exercises 41-48.

**11.4.4. Homework Set #4**

- Read 11.4 (pages 792-797).
- Do exercises on pages 798-799:
  1, 3, 5, 7, 9, 11, 12, 13, 15, 17, 19, 27, 29, 31, 33, 35, 37, 41, 43, 45, 47, 53, 54, 55, 56, 57, 58, 61