# Lecture 3

# 11.3. The Dot Product of Two Vectors

**Goals:** (1) Use properties of the dot product of two vectors.

(2) Find the angle between two vectors using the dot product.

(3) Find the direction cosines of a vector in space.

(4) Find the projection of a vector onto another vector.

(5) Use vectors to find the work done by a constant force.

Questions:

- Recall: what are the operations of vectors we have learned so far? Notice that the sum is a vector, so is the scalar multiplication.
- What is the Law of Cosines?

### 11.3.1. The dot product of two vectors

(1) Definitions:

- Two-dim vectors: $\langle u_1, u_2\rangle \cdot \langle v_1, v_2\rangle$, the ***dot product*** of two vectors $\langle u_1, u_2\rangle$ and $\langle v_1, v_2\rangle$, is $u_1v_1 + u_2v_2$
- Three-dim vectors: $\langle u_1, u_2, u_3\rangle \cdot \langle v_1, v_2, v_3\rangle$, the ***dot product*** of two vectors $\langle u_1, u_2, u_3\rangle$ and $\langle v_1, v_2, v_3\rangle$, is $u_1v_1 + u_2v_2 + u_3v_3$
- Notes:
  - The dot product of two vectors… not of a number and a vector!
  - The dot product is always a scalar (number), not a vector!
- Advanced questions…
  What is an *n*-dim vector?
  What is the dot product of two *n*-dim vectors?

(2) Some properties of the dot product

- Commutative: $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
- Distributive: $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$
- Associative: $c(\vec{u}\cdot\vec{v}) = (c\vec{u})\cdot\vec{v} = \vec{u}\cdot(c\vec{v})$
- Zero-multiplication: $\vec{0}\cdot\vec{v} = 0$. What about $\vec{v}\cdot\vec{0}$?
- Self-multiplication: $\vec{v}\cdot\vec{v} = \|\vec{v}\|^2$. Why? What is $\|\vec{v}\|$?

(3) Example 1: Finding dot products (page 784)

- Try exercises 9, 10

### 11.3.2. Angle between two vectors

(1) Definitions (see Figure 11.24):

- The angle $\theta$ between two (nonzero) vectors is between 0 and $\pi$. That is, $0 \le \theta \le \pi$.

- $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$
- $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$
- $\vec{u}$ and $\vec{v}$ are called ***orthogonal*** if $\vec{u}\cdot\vec{v} = 0$.

  Note: Two nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\theta = \frac{\pi}{2}$.

  Note: $\vec{0}$ is orthogonal to every vector!

(2) Example 2 (page 785)
Finding the angle between two vectors.

- Try exercises 11-26.

**11.3.3. Direction cosines**

See Figure 11.26 (page 786)

(1) Definitions: Given $\vec{v} = \langle v_1, v_2, v_3\rangle$.

- Direction angles: $\alpha, \beta, \gamma$
- Direction cosines:
  $\cos\alpha = \frac{v_1}{\|\vec{v}\|}, \cos\beta = \frac{v_2}{\|\vec{v}\|}, \cos\gamma = \frac{v_3}{\|\vec{v}\|}$

(2) Property:
$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

(3) Example 3 (page 786)
Finding direction angles

- Try exercises 31-38.

**11.3.4. Decomposition, Projections & Vector components**

See Figure 11.29 (page 787)

(1) Definition: Let $\vec{u}$ and $\vec{v}$ be nonzero vectors.

- If $\vec{u} = \vec{w}_1 + \vec{w}_2$, where $\vec{w}_1$ and $\vec{w}_2$ are orthogonal nonzero vectors, then we say that $\vec{u}$ is ***decomposed*** into $\vec{w}_1$ and $\vec{w}_2$. $\vec{w}_1$ and $\vec{w}_2$ are called ***vector components***.
- Furthermore, if $\vec{w}_1$ is parallel to $\vec{v}$, then $\vec{w}_1$ is called the ***projection*** of $\vec{u}$ onto $\vec{v}$, denoted by $\vec{w}_1 = \mathrm{proj}_{\vec{v}}\,\vec{u}$. And $\mathrm{proj}_{\vec{v}}\,\vec{u}$ is also called the vector component of $\vec{u}$ ***along*** $\vec{v}$.
- $\vec{w}_2 = \vec{u} - \mathrm{proj}_{\vec{v}}\,\vec{u}$ is called the vector component of $\vec{u}$ ***orthogonal*** to $\vec{v}$.

(2) How to find $\mathrm{proj}_{\vec{v}}\,\vec{u}$?
Answer: $\mathrm{proj}_{\vec{v}}\,\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}$.

(3) Examples 4 & 5 (pages 787-788)
Decomposing a vector into vector components.

- Try exercises 43-50.

**11.3.5. Applications**

(1) Example 6 (page 788)
Finding a force.

- Try exercise 71.

(2) Example 7 (pageb789)
Finding work.

- Try exercises 73-76.

### 11.3.6. Homework Set #3

- Read 11.3 (pages 783-789).
- Do exercises on pages 789-791:
  1, 3, 5, 7, 9, 11, 12, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 41, 43, 45, 47, 49, 65, 67, 69, 71, 73, 75, 76, 77, 78