# Review for Final Exam

(1) Problems from Chapter 11 will not be on the final exam. However, in order to understand Chapters 12, 13, 14, and 15, you should be familiar with the concepts in Chapter 11.

(2) (12.1) Limits of vector-valued functions.
(a) $\lim_{t\to a} \vec{r}(t) = [\lim_{t\to a} f(t)]\vec{\imath} + [\lim_{t\to a} g(t)]\vec{\jmath}$ , provided that the component limits exist.
(b) $\lim_{t\to a} \vec{r}(t) = [\lim_{t\to a} f(t)]\vec{\imath} + [\lim_{t\to a} g(t)]\vec{\jmath} + [\lim_{t\to a} h(t)]\vec{k}$ , provided that the component limits exist.

(3) (12.1) Continuity of vector-valued functions.
A vector-valued function $\vec{r}$ is continuous at the point given by $t = a$ if the limit $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$.

(4) (12.2) Differentiate a vector-valued function.
Differentiation of vector-value based functions can be done on a *component-by-component basis*. That is, let $\vec{r}(t) = f(t)\vec{\imath} + g(t)\vec{\jmath} + h(t)\vec{k}$. Then

$$\vec{r}'(t) = f'(t)\vec{\imath} + g'(t)\vec{\jmath} + h'(t)\vec{k}$$

(5) (12.2) Integrate a vector-valued function.
Integration of vector-value based functions can also be done on a *component-by-component basis*. That is, let $\vec{r}(t) = f(t)\vec{\imath} + g(t)\vec{\jmath} + h(t)\vec{k}$. Then

$$\int \vec{r}(t)dt = \left[\int f(t)dt\right]\vec{\imath} + \left[\int g(t)dt\right]\vec{\jmath} + \left[\int h(t)dt\right]\vec{k}$$

$$\int_a^b \vec{r}(t)dt = \left[\int_a^b f(t)dt\right]\vec{\imath} + \left[\int_a^b g(t)dt\right]\vec{\jmath} + \left[\int_a^b h(t)dt\right]\vec{k}$$

(6) (12.3) Describe the velocity and acceleration associated with a vector-valued function.
Let $\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath}$ be a vector-valued function where $x(t)$ and $y(t)$ are twice-differentiable functions of $t$. Then
Velocity = $\vec{v}(t) = \vec{r}'(t) = x'(t)\vec{\imath} + y'(t)\vec{\jmath}$
Speed = $\|\vec{v}(t)\| = \|\vec{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$
Acceleration = $\vec{a}(t) = \vec{r}''(t) = x''(t)\vec{\imath} + y''(t)\vec{\jmath}$

(7) (12.3) Use a vector-valued function to analyze projectile motion.
The position function for a projectile is:

$$\vec{r}(t) = (v_0 \cos\theta)t\vec{\imath} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\vec{\jmath}$$

where $h$ is the initial height, $v_0$ is the initial speed, $\theta$ is the initial angle of elevation, and $g = 32$ feet per second per second (or $9.81$ meters per second per second) is the *gravitational constant*.

(8) (12.4) Find a unit tangent vector at a point on space curve.
Let $C$ be a smooth curve represented by $\vec{r}$ on an open interval. The unit tangent vector at $t$ is defined to be

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}, \vec{r}'(t) \neq \vec{0}$$

(9) (12.4) Find a unit normal vector at a point on space curve.
Let $C$ be a smooth curve represented by $\vec{r}$ on an open interval. The principal unit normal vector at $t$ is defined to be

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}, \vec{T}'(t) \neq \vec{0}$$

(10) (12.4) Find the tangential and normal components of acceleration.
If $\vec{N}(t)$ exists (which implies that $\vec{T}(t)$ also exists), then

$$\vec{a}(t) = a_{\vec{T}}\vec{T}(t) + a_{\vec{N}}\vec{N}(t)$$

where

$$a_{\vec{T}} = \|\vec{v}\|' = \vec{a} \cdot \vec{T} = \frac{\vec{v} \cdot \vec{a}}{\|\vec{v}\|}$$
$$a_{\vec{N}} = \|\vec{v}\|\|\vec{T}'\| = \vec{a} \cdot \vec{N} = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|} = \sqrt{\|\vec{a}\|^2 - {a_{\vec{T}}}^2}$$

(11) (12.5) Find the arc length of a space curve.
If $C$ is a smooth curve (in space) given by $\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$, on an interval $[a, b]$, then the arc length of $C$ on the interval is:

$$s = \int_a^b \|\vec{r}'(t)\|\, dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$

(12) (12.5) Use the arc length parameter to describe a plane curve or space curve.
Let $C$ be a smooth curve given by $\vec{r}(t)$ on a closed interval $[a, b]$. For $a \leq t \leq b$, the arc length *function* is given by:

$$s(t) = \int_a^t \|\vec{r}'(w)\|\, dw = \int_a^t \sqrt{[x'(w)]^2 + [y'(w)]^2 + [z'(w)]^2}\, dw$$

If we choose $s$ as the parameter, that is, $\vec{r}(s) = x(s)\vec{\imath} + y(s)\vec{\jmath} + z(s)\vec{k}$, then $\|\vec{r}'(s)\| = 1$. On the other hand, if $\|\vec{r}'(t)\| = \frac{ds}{dt} = 1$, then $t$ must be the arc length parameter. Therefore, the parameter $t$ is the arc length parameter if and only if $\|\vec{r}'(t)\| = 1$.

(13) (12.5) Find the curvature of a curve at a point on the curve.
(a) Let $C$ be a smooth curve given by $\vec{r}(s)$ where $s$ is the arc length parameter. The curvature $K$ at $s$ is given by

$$K = \left\|\frac{d\vec{T}}{ds}\right\| = \|\vec{T}'(s)\|$$

(b) In general, if the parameter is $t$ (not necessary to be $s$), then the curvature at $t$ is given by

$$K = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

(c) In particular, if $C$ is given by a regular (rectangular coordinates) function $y = f(x)$, then the curvature at $P(x, y)$ is given by

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}}$$

(14) (12.5) Relationship among acceleration, speed, and curvature.

$$\vec{a}(t) = \frac{d^2s}{dt^2}\vec{T} + K\left(\frac{ds}{dt}\right)^2 \vec{N}$$

(15) (13.1) Sketch the graph of a function of two variables.
Use traces in planes parallel to the coordinate planes.

(16) (13.1) Sketch level curves for a function of two variables.
Another way to visualize a function $z = f(x, y)$ is to use a *scalar field* that is characterized by *level curves* $f(x, y) = c$ (contour lines, or equipotential lines).

(17) (13.1) Sketch level surfaces for a function of three variables.
The concept of a level curve can be extended by one dimension to define a level surface. Given a function $z = f(x, y, z)$, the graph of the equation $f(x, y, z) = c$ is called a *level surface*.

(18) (13.2) Understand and use the definition of the limit of a function of two variables.
How to determine whether the limit exists? All paths/directions of $(x, y)$ approaching $(x_0, y_0)$ should lead to the same limit.

(19) (13.2) Extend the concept of continuity to a function of two variables.
(a) A function $z = f(x, y)$ is said to be continuous at $(x_0, y_0)$ in an open region $R$ if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

(b) Removable and nonremovable discontinuity:
If $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists, but $\neq f(x_0, y_0)$, or $f(x_0, y_0)$ is not defined, then it is removable in both cases.
If $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist, then it is not removable.

(20) (13.3) Find and use partial derivatives of a function of two variables.
The first partial derivatives of $z = f(x, y)$ with respect to $x$ and $y$ are the functions $f_x$ and $f_y$ defined by the following limits:

$$f_x(x, y) = \lim_{\Delta x\to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$
$$f_y(x, y) = \lim_{\Delta y\to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

provided that the limits exist.

(21) (13.3) Find and use partial derivatives of a function of three or more variables.
Let $w = f(x, y, z)$ be a function in three variables. Then there will be three partial derivatives:

$$\frac{\partial w}{\partial x} = f_x(x, y, z) = \lim_{\Delta x\to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$
$$\frac{\partial w}{\partial y} = f_y(x, y, z) = \lim_{\Delta y\to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$
$$\frac{\partial w}{\partial z} = f_z(x, y, z) = \lim_{\Delta z\to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

(22) (13.3) Find higher-order partial derivatives of a function of two or three variables.
2nd-order partial derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

(23) (13.4) Use the differential as an approximation.
$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$ and $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$.
$\Delta z \approx dz$ is called a linear approximation.

(24) (13.5) Chain rules with one variable for functions of several variables.
Let $w = f(x, y)$, $x = g(t)$, and $y = h(t)$ be differentiable functions. Then $w = f\big(g(t), h(t)\big)$ is a differentiable function in one variable $t$, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

Note: This formula can be extended to 3 or more variables.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

(25) (13.5) Chain rules with two independent variables.
Let $w = f(x, y)$, $x = g(s, t)$, and $y = h(s, t)$ be differentiable functions and all of the partial derivatives exist. Then $w = f\big(g(s, t), h(s, t)\big)$ is a differentiable function in two variables $s, t$, and

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}, \text{ and } \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

Note: This formula can be extended to 3 or more variables.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}, \text{ and } \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

(26) (13.5) Find partial derivatives implicitly.
(a) If $F(x, y) = 0$ is an implicit function, then the <u>derivative</u> $\frac{dy}{dx}$ can be found implicitly:

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}, \text{ where } F_y(x, y) \neq 0$$

(b) If $F(x, y, z) = 0$ is an implicit function, then the two <u>partial derivatives</u> can be found implicitly:

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}, \text{ where } F_z(x, y, z) \neq 0$$

(27) (13.6) Find and use directional derivatives of a function of two variables.
Let $f$ be a function of two variables $x, y$ and let $\vec{u} = \cos\theta\, \vec{i} + \sin\theta\, \vec{j}$ be a unit vector. Then

$$D_{\vec{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

(28) (13.6) Find the gradient of a function of two variables.
Let $z = f(x, y)$ be a function whose partial derivatives exist. Then the gradient of $f$, denoted by $\nabla f(x, y)$, or grad $f(x, y)$, is the vector-valued function:

$$\nabla f(x, y) = f_x(x, y)\, \vec{i} + f_y(x, y)\, \vec{j}$$

(29) (13.6) Find directional derivatives and gradients of functions of three variables.
Let $f$ be a function of three variables $x, y, z$ and let $\vec{u} = a\vec{\imath} + b\vec{\jmath} + c\vec{k}$ be a unit vector. Then

$$D_{\vec{u}}f(x,y,z) = af_x(x,y,z) + bf_y(x,y,z) + cf_z(x,y,z)$$

The gradient is:

$$\nabla f(x,y,z) = f_x(x,y,z)\,\vec{\imath} + f_y(x,y,z)\,\vec{\jmath} + f_z(x,y,z)\,\vec{k}$$

(30) (13.7) Find equations of tangent planes and normal lines to surfaces.
Use $\nabla F(x_0, y_0, z_0)$ as the directional vector!

(31) (13.7) Find the angle of inclination of a plane in space.
Let $\vec{n}$ be a normal vector of a plane, and $\vec{k} = \langle 0, 0, 1\rangle$. Then the angle of inclination of a plane is given by:

$$\cos\theta = \frac{|\vec{n}\cdot\vec{k}|}{\|\vec{n}\|}$$

where $\vec{n}$ can be $\nabla F(x,y,z) = F_x(x,y,z)\,\vec{\imath} + F_y(x,y,z)\,\vec{\jmath} + F_z(x,y,z)\,\vec{k}$. In particular, if $z = f(x,y)$, then $F(x,y,z) = f(x,y) - z = 0$, and $\nabla F(x,y,z) = f_x(x,y)\,\vec{\imath} + f_y(x,y)\,\vec{\jmath} - \vec{k}$.

(32) (13.8) Find relative extrema of a function of two variables.
$(x_0, y_0)$ is called a critical point if $\nabla f(x_0, y_0) = \vec{0}$ (that is, both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$) or at least one of them does not exist. If $f$ has a relative extremum at $(x_0, y_0)$ on an open region $R$, then $(x_0, y_0)$ must be a critical point of $f$.

(33) (13.8) Use the Second Partials Test to find relative extrema of a function of two variables.
Let $f$ have continuous second partial derivatives on an open region containing a point $(a,b)$ such that $\nabla f(a,b) = 0$ (critical point). Let

$$d = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

(a) If $d > 0$ and $f_{xx}(a,b) > 0$, then $f$ has relative minimum at $(a,b)$
(b) If $d > 0$ and $f_{xx}(a,b) < 0$, then $f$ has relative maximum at $(a,b)$
(c) If $d < 0$, then $\left(a, b, f(a,b)\right)$ is a saddle point.
(d) If $d = 0$, then the test fails (it is inconclusive).

(34) (13.9) Solve optimization problems involving functions of several variables.
Find relative extrema by taking partial derivatives.

(35) (13.10) Understand the Method of Lagrange Multipliers.
Lagrange's Theorem: Let $f$ and $g$ have continuous first partial derivatives such that $f$ has an extremum at a point $(x_0, y_0)$ on the smooth constraint curve $g(x,y) = c$. If $\nabla g(x_0, y_0) \neq \vec{0}$, then there is a real number $\lambda$ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

(36) (13.10) Use Lagrange multipliers to solve constrained optimization problems.
Method of Lagrange Multipliers (one constraint):
Let $f$ and $g$ have continuous first partial derivatives such that $f$ has an extremum at a point $(x_0, y_0)$ on the smooth constraint curve $g(x,y) = c$. The following steps help us find the min or max value of $f$.
Step 1: Solve the following system of equations:

$$f_x(x,y) = \lambda g_x(x,y)$$
$$f_y(x,y) = \lambda g_y(x,y)$$
$$g(x,y) = c$$

Step 2: Evaluate at each solution. The largest is max, and the smallest is the min.

Note: step one above can be extended to three variables case:

$$f_x(x,y,z) = \lambda g_x(x,y,z)$$
$$f_y(x,y,z) = \lambda g_y(x,y,z)$$
$$f_z(x,y,z) = \lambda g_z(x,y,z)$$
$$g(x,y,z) = c$$

(37) (13.10) Use the Method of Lagrange Multipliers with two constraints.

Method of Lagrange Multipliers (two constraints):

With the same setting as in one variable case, except for two constraints, $g$ and $h$, then $\nabla f = \lambda \nabla g + \mu \nabla h$ will lead to potential extrema.

Step 1: Solve the following system of equations:

$$f_x(x,y,z) = \lambda g_x(x,y,z) + \mu h_x(x,y,z)$$
$$f_y(x,y,z) = \lambda g_y(x,y,z) + \mu h_y(x,y,z)$$
$$f_z(x,y,z) = \lambda g_z(x,y,z) + \mu h_z(x,y,z)$$
$$g(x,y,z) = c_1$$
$$h(x,y,z) = c_2$$

Step 2: Evaluate at each solution. The largest is max, and the smallest is the min.

(38) (14.1) Use an iterated integral to find the area of a plane region.

(a) Vertically simple region:

$$A = \int_a^b [g_2(x) - g_1(x)]\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx$$

(b) Horizontally simple region:

$$A = \int_c^d [h_2(y) - h_1(y)]\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\,dy$$

(39) (14.2) Use a double integral to represent the volume of a solid region.

If $f(x,y) \geq 0$ is integrable on a closed, bounded region $R$ in the $xy$-plane, then the volume of the solid region that lies above $R$ and below the graph of $f$ is defined as the double integral:

$$V = \iint_R f(x,y)dA$$

(40) (14.2) Evaluate a double integral as an iterated integral.

(a) Vertically simple region: if $R$ is defined by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where $g_1, g_2$ are continous on $[a,b]$, then

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dy\,dx$$

(b) Horizontally simple region: if $R$ is defined by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where $h_1, h_2$ are continous on $[c,d]$, then

$$\iint\limits_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dx\,dy$$

(41) (14.3) Write and evaluate double integrals in polar coordinates.
Let $R$ be a plane region consisting of all points $(x,y) = (r\cos\theta, r\sin\theta)$ satisfying the conditions $0 \le g_1(\theta) \le r \le g_2(\theta), \alpha \le \theta \le \beta$ (that is, $r$-simple), where $0 \le (\beta - \alpha) \le 2\pi$. If $g_1$ and $g_2$ are continuous on $[\alpha,\beta]$ and $f$ is continuous on $R$, then

$$\iint\limits_R f(x,y)dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)r dr\,d\theta$$

(42) (14.4) Find the mass of a planar lamina using a double integral.
If $\rho$ is a continuous density function on the lamina corresponding to a plane region $R$, then the mass $m$ of the lamina is given by

$$m = \iint\limits_R \rho(x,y)dA$$

(43) (14.4) Find the center of mass of a planar lamina using double integrals.
Let $\rho$ be a continuous density function on the planar lamina region $R$. The moments of mass $m$ with respect to the $x$- and $y$-axes are

$$M_x = \iint\limits_R y\rho(x,y)dA \quad \text{and} \quad M_y = \iint\limits_R x\rho(x,y)dA$$

And the center of mass is

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

(44) (14.4) Find moments of inertia using double integrals.
Second moments are also called the **moment of inertia** of a lamina about a line:

$$I_x = \iint\limits_R y^2\rho(x,y)dA \quad \text{and} \quad I_y = \iint\limits_R x^2\rho(x,y)dA$$

(45) (14.5) Use double integral to find the area of a surface.
If $f$ and its partial derivatives are continuous on the closed region $R$ in the $xy$-plane, then the area of the surface $S$ given by $z = f(x,y)$ over $R$ is given by

$$\iint\limits_R dS = \iint\limits_R \sqrt{1 + [f_x'(x,y)]^2 + \left[f_y'(x,y)\right]^2}\,dA$$

(46) (14.6) Use a triple integral to find the volume of a solid region.
Fubini's Theorem on triple integrals
Let $f$ be continuous on a solid region $Q$ defined by

$$a \le x \le b, \qquad h_1(x) \le y \le h_2(x), \qquad g_1(x,y) \le z \le g_2(x,y)$$

where $h_1, h_2, g_1,$ and $g_2$ are continuous functions. Then,

$$\iiint\limits_Q f(x,y,z)dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)dz\,dy\,dx.$$

(47) (14.6) Find the center of mass and moments of inertia of a solid region.

(a) If $\rho$ is a continuous density function on a solid region $Q$, the center of mass is given by

$$\bar{x} = \frac{M_{yz}}{m}, \qquad \bar{y} = \frac{M_{zx}}{m}, \qquad \bar{z} = \frac{M_{xy}}{m}$$

where

$$m = \iiint_Q \rho(x, y, z)dV$$

$$M_{yz} = \iiint_Q x\rho(x, y, z)dV$$

$$M_{zx} = \iiint_Q y\rho(x, y, z)dV$$

$$M_{xy} = \iiint_Q z\rho(x, y, z)dV$$

(b) Second moments are also called the moments of inertia of a solid region about $x$-axis, $y$-axis, and $z$-axis, respectively:

$$I_x = \iiint_Q (y^2 + z^2)\rho(x, y, z)dV$$

$$I_y = \iiint_Q (z^2 + x^2)\rho(x, y, z)dV$$

$$I_z = \iiint_Q (x^2 + y^2)\rho(x, y, z)dV$$

(48) (14.7) Write and evaluate a triple integral in cylindrical coordinates.

Note: $dV = rdz\, dr\, d\theta$ for cylindrical coordinates. See Figure 14.62 (p. 1035)

(a) If $Q$ is a $r$-simple, then

$$\iiint_Q f(x, y, z)dV = \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r\cos\theta, r\sin\theta)}^{h_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, rdz\, dr\, d\theta$$

(b) If $Q$ is a $\theta$-simple, then

$$\iiint_Q f(x, y, z)dV = \int_{r_1}^{r_2} \int_{g_1(r)}^{g_2(r)} \int_{h_1(r\cos\theta, r\sin\theta)}^{h_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, rdz\, d\theta\, dr$$

(49) (14.7) Write and evaluate a triple integral in spherical coordinates.

If $Q$ is a spherical block determined by

$\{(\rho, \theta, \phi): \rho_1 \le \rho \le \rho_2, \theta_1 \le \theta \le \theta_2, \phi_1 \le \phi \le \phi_2\}$. Then

$$\iiint_Q f(x,y,z)dV$$
$$= \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

(50) (15.1) Determine whether a vector field is conservative.
Test for conservative vector field in the *plane*:
Let $M$ and $N$ have continuous first partial derivatives on an open disk $R$. The vector field given by $\vec{F}(x,y) = M\vec{\imath} + N\vec{\jmath}$ is conservative if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

(51) (15.1) Find the curl of a vector field.
(a) The curl of vector field $\vec{F}(x,y,z) = M\vec{\imath} + N\vec{\jmath} + P\vec{k}$ is

$$\overrightarrow{\text{curl}}\,\vec{F}(x,y,z) = \nabla\times\vec{F}(x,y,z) =$$
$$\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{\imath} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}.$$

If $\overrightarrow{\text{curl}}\,\vec{F}(x,y,z) = \vec{0}$, then $\vec{F}(x,y,z)$ is said to be irrotational.
(b) Test for conservative vector field in *space*:
Let $M$, $N$ and $P$ have continuous first partial derivatives on an open sphere $Q$ in space. The vector field given by $\vec{F}(x,y,z) = M\vec{\imath} + N\vec{\jmath} + P\vec{k}$ is conservative if and only if $\overrightarrow{\text{curl}}\,\vec{F}(x,y,z) = \vec{0}$. That is,

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

(52) (15.1) Find the divergence of a vector field.
The divergence of a vector field (in plane) $\vec{F}(x,y) = M\vec{\imath} + N\vec{\jmath}$ is:

$$\text{div}\,\vec{F}(x,y) = \nabla\cdot\vec{F}(x,y) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

The divergence of a vector field (in space) $\vec{F}(x,y,z) = M\vec{\imath} + N\vec{\jmath} + P\vec{k}$ is:

$$\text{div}\,\vec{F}(x,y,z) = \nabla\cdot\vec{F}(x,y,z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

If $\text{div}\,\vec{F} = 0$, that is, the sum of partial derivatives equals 0, then $\vec{F}$ is said to be divergence free.

(53) (15.2) Write and evaluate a line integral.
Let $f$ be continuous in a region containing a smooth curve $C$. If $C$ is given by $\vec{r}(t)$, where $a \le t \le b$, then $ds = \|\vec{r}\,'(t)\|dt$.

$$\int_C f(x,y)\,ds = \int_a^b f\big(x(t),y(t)\big)\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$
$$\int_C f(x,y,z)\,ds = \int_a^b f\big(x(t),y(t),z(t)\big)\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt$$

If $C$ is a piecewise smooth path composed of smooth curves $C_1, C_2, \dots, C_n$, then

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \cdots + \int_{C_n} f\,ds$$

(54) (15.2) Write and evaluate a line integral of a vector field.
Let $\vec{F}$ be a continuous vector field (also called force field) defined on a smooth curve $C$ given by $\vec{r}(t), a \le t \le b$. The line integral of vector field $\vec{F}$ on $C$ is given by

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F}\big(x(t), y(t), z(t)\big) \cdot \vec{r}'(t)\,dt$$

(55) (15.2) Write and evaluate a line integral in differential form.
The differential form of a line integral of a vector field (in plane) $\vec{F}(x,y) = M\vec{\imath} + N\vec{\jmath}$ along a curve $C$ given by $\vec{r}(t), a \le t \le b$ is:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C Mdx + Ndy$$

The differential form of a line integral of a vector field (in space) $\vec{F}(x,y,z) = M\vec{\imath} + N\vec{\jmath} + P\vec{k}$ along a curve $C$ given by $\vec{r}(t), a \le t \le b$ is:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C Mdx + Ndy + Pdz$$

(56) (15.3) Understand and use the Fundamental Theorem of Line Integrals.
Let $C$ be a piecewise smooth curve lying in an open region $R$ and given by

$$\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath}, \qquad a \le t \le b.$$

If $\vec{F}(x,y) = M\vec{\imath} + N\vec{\jmath}$ is *conservative* in $R$, and $M$ and $N$ are continuous in $R$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f\big(x(b), y(b)\big) - f\big(x(a), y(a)\big)$$

where $f$ is a potential function of $\vec{F}$. That is, $\vec{F}(x,y) = \nabla f(x,y)$.

(57) (15.3) Understand the concept of independence of path.
Let $\vec{F}(x,y,z) = M\vec{\imath} + N\vec{\jmath} + P\vec{k}$ have continuous first partial derivatives in an open connected region $R$, and let $C$ be a piecewise smooth curve in $R$. The following conditions are equivalent:

(a) $\vec{F}$ is conservative.

(b) $\int_C \vec{F}(x,y,z) \cdot d\vec{r}$ is independent of path.

(c) $\int_C \vec{F}(x,y,z) \cdot d\vec{r} = 0$ for every closed curve $C$ in $R$.

(58) (15.4) Use Green's Theorem to evaluate a line integral.
Let $R$ be a simply connected region with a piecewise smooth boundary $C$, oriented counterclockwise (that is, $C$ traversed once so that the region $R$ always lies to the left). If $M$ and $N$ have continuous partial derivatives in an open region containing $R$, then

$$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

(59) (15.4) Use alternative forms of Green's Theorem.
First alternative form:

$$\int_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R \left(\overrightarrow{\text{curl}}\,\vec{F}\right)\cdot \vec{k}\, dA$$

Second alternative form:

$$\int_C \vec{F}\cdot\vec{N}\, ds = \iint_R \text{div}\,\vec{F} dA$$

(60) (15.5) Find a normal vector and a tangent plane to a parametric surface.
Let $S$ be a smooth parametric surface

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

defined over an open region $D$ in the $uv$-plane. Let $(u_0, v_0)$ be a point in $D$. A normal vector at the point

$$(x_0, y_0, z_0) = \big(x(u_0,v_0), y(u_0,v_0), z(u_0,v_0)\big)$$

is given by

$$\vec{N} = \vec{r}_u^{\,\prime}(u_0,v_0) \times \vec{r}_v^{\,\prime}(u_0,v_0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

(61) (15.5) Find the area of a parametric surface.
Let $S$ be a smooth parametric surface

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

defined over an open region $D$ in the $uv$-plane. If each point on the surface $S$ corresponds to exactly one point in the domain $D$, then the surface area of $S$ is given by

$$\text{Surface area} = \iint_S dS = \iint_D \|\vec{N}\| dA = \iint_D \|\vec{r}_u^{\,\prime} \times \vec{r}_v^{\,\prime}\| dA$$

where

$$\vec{r}_u^{\,\prime} = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}, \text{ and } \vec{r}_v^{\,\prime} = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}.$$

(62) (15.6) Evaluate a surface integral as a double integral.
Let $S$ be a surface with equation $z = g(x,y)$ and let $R$ be its projection onto the $xy$-plane. If $g, g_x', g_y'$ are continuous on $R$ and $f$ is continuous on $S$, then the surface integral of $f$ over $S$ is

$$\iint_S f(x,y,z) dS = \iint_R f\big(x,y,g(x,y)\big)\sqrt{1 + [g_x'(x,y)]^2 + \left[g_y'(x,y)\right]^2}\, dA.$$

(63) (15.6) Evaluate a surface integral for a parametric surface.
Let $S$ be a smooth parametric surface given by vector-valued function

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

defined over an open region $D$ in the $uv$-plane. The surface integral of $f$ over $S$ is

$$\iint_S f(x,y,z)dS = \iint_D f\big(x(u,v),y(u,v),z(u,v)\big)\|\vec{r}_u'(u,v)\times\vec{r}_v'(u,v)\|dA$$

(64) (15.6) Understand the concept of a flux integral.

Definition:

Let $\vec{F}(x,y,z) = M\vec{\imath} + N\vec{\jmath} + P\vec{k}$ be a vector field, where $M, N, P$ have continuous first partial derivatives on the oriented surface $S$ with the unit normal vector $\vec{N}_1$. The flux integral of $\vec{F}$ across $S$ is given by

$$\iint_S \vec{F}\cdot\vec{N}_1 dS$$

Geometrically, a flux integral is the surface integral over $S$ of the *normal component* of $\vec{F}$.

(a) If $S$ is a surface with equation $z = g(x,y)$ and $R$ is its projection onto the $xy$-plane, then

$$\iint_S \vec{F}\cdot\vec{N}_1\, dS = \iint_R \vec{F}\cdot\left[-g_x'(x,y)\vec{\imath} - g_y'(x,y)\vec{\jmath} + \vec{k}\right] dA$$

(b) If $S$ is an oriented surface given by vector-valued function $\vec{r}(u,v) = x(u,v)\vec{\imath} + y(u,v)\vec{\jmath} + z(u,v)\vec{k}$ defined over a region $D$ in the $uv$-plane, then

$$\iint_S \vec{F}\cdot\vec{N}_1\, dS = \iint_D \vec{F}\cdot\big(r_u'(u,v)\times r_v'(u,v)\big)\, dA$$

(65) (15.7) Understand and use the Divergence Theorem.

Let $Q$ be a solid region bounded by a *closed* surface $S$ oriented by a unit normal vector directed outward from $Q$. If $\vec{F}$ is a vector field whose component functions have continuous partial derivatives in $Q$, then

$$\iint_S \vec{F}\cdot\vec{N}_1\, dS = \iiint_Q \operatorname{div}\vec{F}\, dV$$

(66) (15.8) Understand and use Stokes' Theorem.

Let $S$ be an oriented surface with unit normal vector $\vec{N}_1$, bounded by a piecewise smooth simple closed curve $C$. If $\vec{F}$ is a vector field whose component functions have continuous partial derivatives on an open region containing $S$ and $C$, then

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \left(\overrightarrow{\text{curl}}\,\vec{F}\right)\cdot\vec{N}_1\, dS$$

(67) (15.8) Use curl to analyze the motion of rotating liquid.

The formula $\int_C \vec{F}\cdot d\vec{r} = \iint_S \left(\overrightarrow{\text{curl}}\,\vec{F}\right)\cdot\vec{N}_1\, dS$ says that the collective measure of this *rotational* tendency taken over the entire surface $S$ (surface integral) is equal to the tendency of a fluid to *circulate* around the boundary $C$ (line integral).