Chapter 5: Probability

In Chapter 5, we step away from data for a while. We take a look at a new topic for us - **probability**. Most of us have an idea already of what probability is, but we'll spend quite a while exploring different probability experiments (like rolling two dice) and investigating the different outcomes.

Source: stock.xchng

We'll learn several different rules, ranging from the probability that at least one of two events occurs in Section 5.2 (the Addition Rule), to the probability that both occur in Section 5.3 (the Multiplication Rule), to the probability that one occurs if we know the first has already occurred in Section 5.4 (conditional probability).

In Section 5.5, we learn some new counting techniques that'll help us answer questions like "How many 4-digit garage door codes are possible if digits can't be repeated once used?"

Source: Sears

Section 5.6 brings it all together, and helps you choose which strategy to apply.

If you're ready to begin, just click on the "start" link below, or one of the section links on the left.

Section 5.1: Probability Rules

Objectives

By the end of this lesson, you will be able to...

1. apply the rules of probability
2. compute and interpret probabilities using the empirical method
3. compute and interpret probabilities using the classical method
4. recognize and interpret subjective probabilities

Probability

So what is probability? Most of us already have an idea. We all know that the probability of heads when flipping a fair coin is 1/2, but what does that mean?

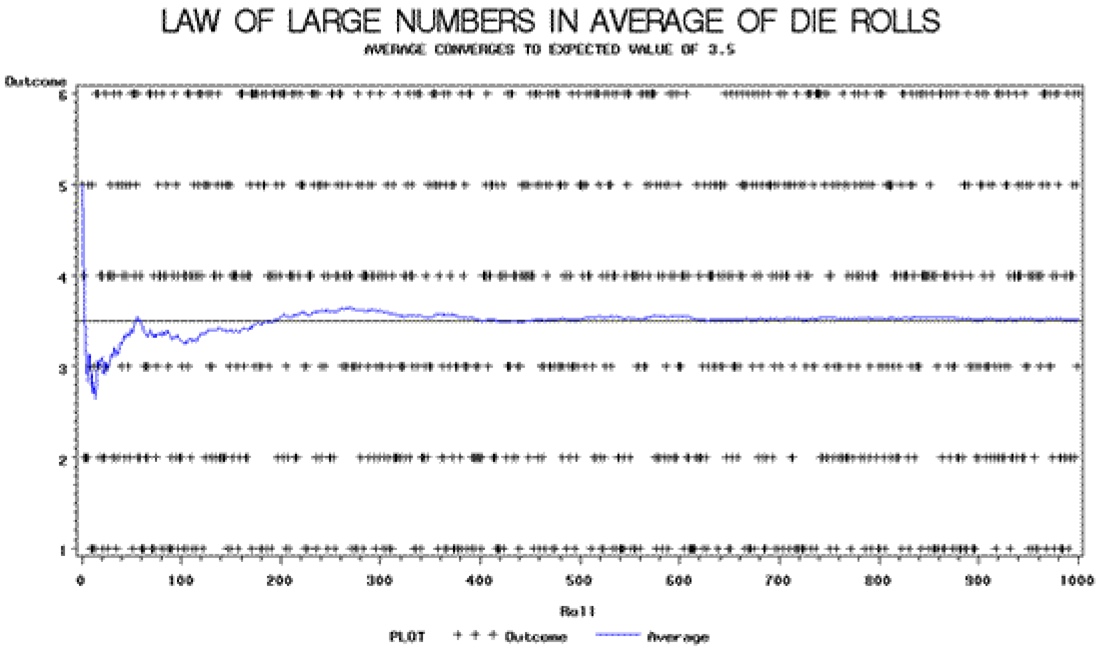
- One out of every two flips will be heads?
- If we have two heads in a row, the next two must be tails?
- Exactly 50 out of every 100 flips will be heads?

In fact, none of these are correct!

> In general, **probability** is a measure of the likelihood of some **outcome**. We use it not to describe what will happen in one particular **event**, but rather, what the long-term proportion that outcome will occur.

So the key here isn't that 1 out of *every* 2 coin flips, or 50 out of *every* 100 coin flips will be heads, but that over the long term, about 1/2 will be.

This concept is called the *Law of Large Numbers*. The image below is from Wikipedia, and shows the idea. It's demonstrating rolling a fair 6-sided die, and calculating the average number. We know that all 6 are equally likely, so the average should be (1+2+3+4+5+6)/6 = 3.5. From the image, we can see that while it isn't 3.5 initially, it does *tend* toward that as the number of rolls increases.



Note: This image is in the public domain and is copyright free.

Probability Experiments

There are some key terms we should outline before going much further.

In probability, an **experiment** is any process where the results are uncertain. We call the **sample space**, S, the collection of all possible outcomes. A probability **event** is any collection of outcomes from the experiment.

Example 1

Suppose we have a family with three children, and we consider the sex of those three children. If we let B represent a boy and G represent a girl, here is the sample space:

S = { BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG}

There are thus 8 *outcomes* in this experiment.

One possible event might be:

E = the family has exactly two girls = { BGG, GBG, GGB }

Example 2

Suppose a fair six-sided die is rolled twice.

What is the sample space?

[reveal answer]

Source: stock.xchng

If we define the event E = the sum of the two dice is more than 10, what outcomes are in E?

[reveal answer]

Unusual Events

Suppose we consider the previous example about rolling two dice. The probability of having the sum of the two dice be more than 10 would be 3/36 or 1/12. Is this unusual? On average, it will occur about 1 in 12 times. Is that unusual enough? We have to be careful when we characterize an event as *unusual*.

> Typically, we say that an event with a probability less than 5% is unusual, but this isn't a hard cutoff. It depends on the context.

Suppose we're planning on making a decision one way, unless the probability of a particularly "unusual" event is too high.

One example might be the jury in a capital case, punishable by death. In this example, jurors need to be sure "beyond a reasonable doubt" that the defendant is guilty. If they decide to convict if they're 95% sure, this means that the "unusual" event that they're wrong has a probability of 5%. If you're that defendant, that's definitely not "unusual" enough!

On the other hand, suppose we're planning a picnic on a nice summer day. If the risk of a rain shower isn't too high, we'll plan on the picnic. In this case, we might set our cutoff at 20% - anything less than that is too unusual (or unlikely) to happen, so we'll risk it.

Calculating Probabilities Using the Empirical Method

There are two primary methods for calculating probabilities The first is to simply look at what has happened in the past and assume the probability is the same as the relative frequency of that particular outcome. This is called the **empirical probability** of that event.

$$P(E) \approx \text{relative frequency of } E = \frac{\text{frequency of } E}{\text{total number of trials}}$$

Example 3

During the Spring semester of 2008, 33 out of 60 students from two sections of Mth096 Basic Algebra at ECC were successful. (Successful here is earning a C or better.)

If we define E = a Mth096 student is successful, P(E) ≈ 33/60 = 0.55.

Example 4

Suppose we give a survey to 52 Basic Algebra students, asking them to rate various statements from Strongly Agree to Strongly Disagree. If 18 of the students responded with either Agree or Strongly Agree to the statement "I enjoy math.", what is the probability that a randomly selected Basic Algebra student will enjoy math?

[reveal answer]

Computing Probabilities Using the Classical Method

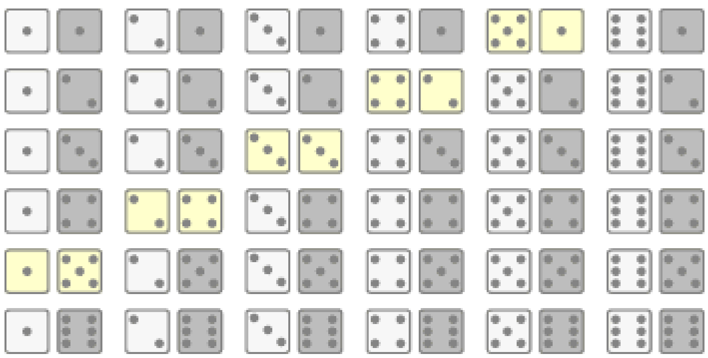
The second primary method for calculating probabilities is the **classical method**. The key for this method is to assume that *all outcomes are equally likely*. This is how we know the probability of rolling a 6 on a fair six-sided die is 1/6, because we assume all of the outcomes (1, 2, 3, 4, 5, and 6) are equally likely.

$$P(E) = \frac{\text{number of ways E can occur}}{\text{total number of possible outcomes}} = \frac{N(E)}{N(S)}$$

Example 5

Let's consider again the probability experiment from Example 2 - rolling two fair six-sided dice.

Let the event E = the sum of the two dice is 6.



Find P(E).

[reveal answer]

Section 5.2: The Addition Rule and Complements

Objectives

By the end of this lesson, you will be able to...

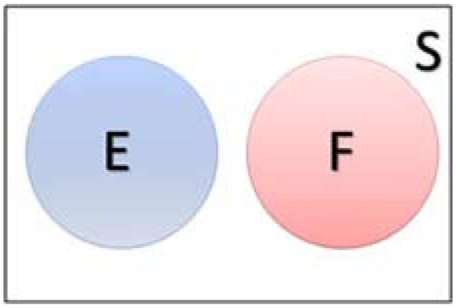
1. use the Addition Rule for disjoint events
2. use the General Addition Rule
3. use the Complement Rule

The Additional Rule for Disjoint Events

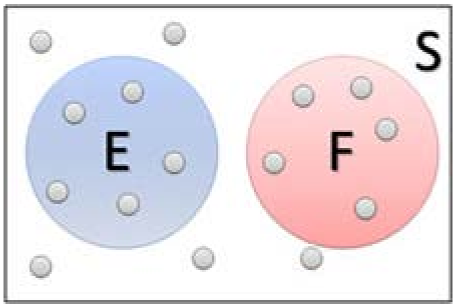
We're going to have quite a few rules in this chapter about probability, but we'll start small. The first situation we want to look at is when two events have no outcomes in common. We call events like this **disjoint events**.

Two events are **disjoint** if they have no outcomes in common. (Also commonly known as **mutually exclusive** events.)

Back in 1881, John Venn developed a great way to visualize sets. As is often the case in mathematics, the diagrams took on his name and have since taken on his name - Venn diagrams. Because events are sets of outcomes, it works well to visualize probability as well. Here's an example of a Venn diagram showing two disjoint outcomes, E and F.



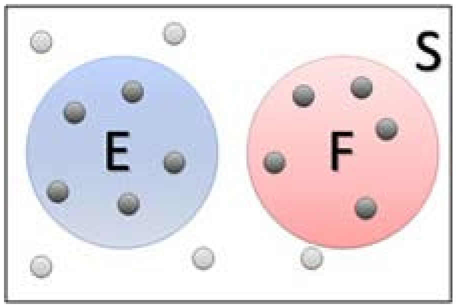
Let's continue this a little further and put points on the chart like this -  - to indicate outcomes.



Looking at the picture, we can clearly see that P(E) = 5/15 = 1/3, since there are 5 outcomes in E, and 15 total outcomes. Similarly, P(F) also is 1/3.

Next, we want to consider all of the events that are in either E or F. In probability, we call that event **E or F**. So in our example, P(E or F) = 10/15 = 2/3.

But, we could just see that from the picture! Just count the dots that are E and add to it the number of dots in F.



In general, we can create a rule. We'll call it...

The Addition Rule for Disjoint Events

If E and F are disjoint (mutually exclusive) events, then

P(E or F) = P(E) + P(F)

Example 1

OK - time for an example. Let's use the example from last section about the family with three children, and let's define the following events:

E = the family has exactly two boys
F = the family has exactly one boy

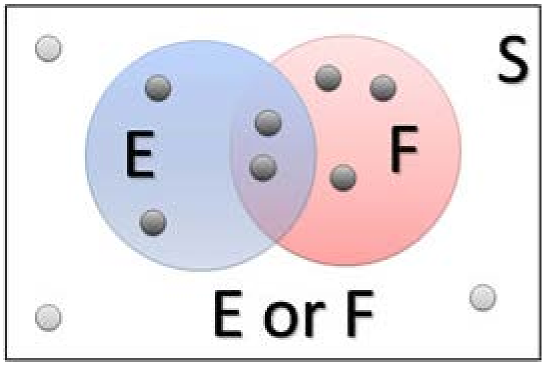
Describe the event "E or F" and find its probability.

[reveal answer]

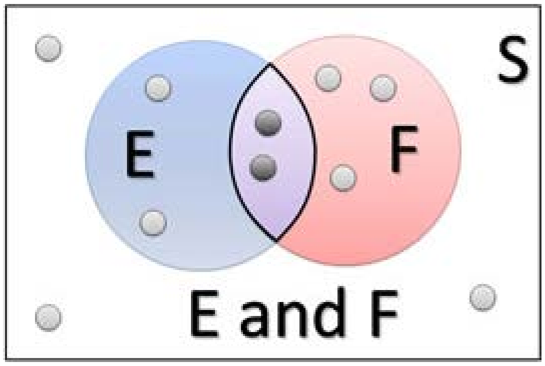
Of course, there are often cases when two events do have outcomes in common, so we'll need a more robust rule for that case.

General Addition Rule

What happens when two events *do* have outcomes in common? Well, let's consider the example below. In this case, P(E) = 4/10 = 2/5, and P(F) = 5/10 = 1/2, but P(E or F) isn't 9/10. Can you see why?



The key here is the two outcomes in the middle where E and F overlap. Officially, we call this region the event **E and F**. It's all the outcomes that are in *both* E *and* F. In our visual example:



In this case, to find P(E or F), we'll need to add up the outcomes in E with the outcomes in F, and then *subtract* the duplicates we counted that are in E and F. We call this the **General Addition Rule**.

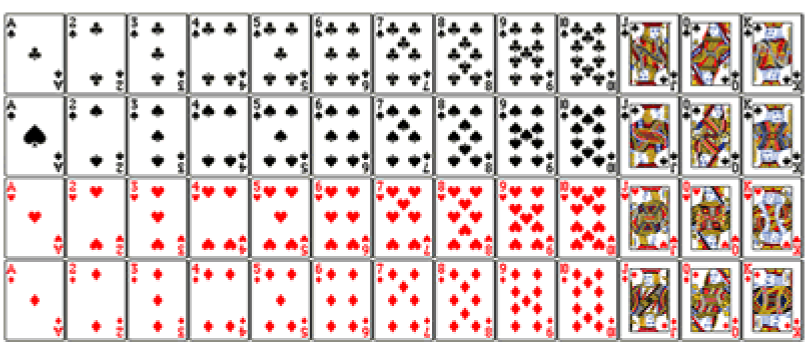
The General Addition Rule

$$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$$

Let's try a couple quick examples.

Example 2

Let's consider a deck of standard playing cards.



Suppose we draw one card at random from the deck and define the following events:

E = the card drawn is an ace
F = the card drawn is a king

Use these definitions to find P(E or F).

[reveal answer]

Example 3

Considering the deck of playing cards, where one is drawn at random. Suppose we define the following events:

F = the card drawn is a king
G = the card drawn is a heart

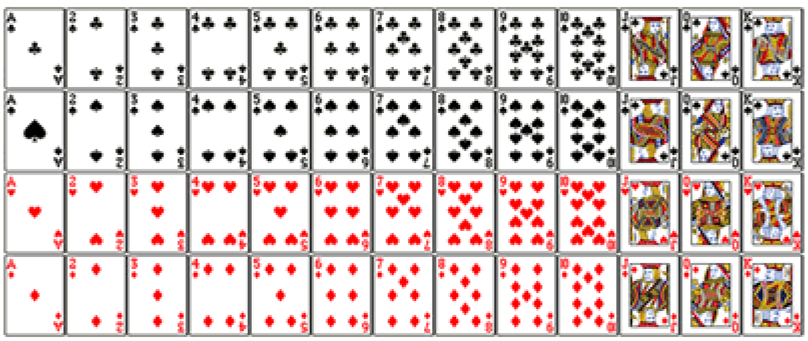
Use these definitions to find P(F or G).

[reveal answer]

So the key idea and the difference between these two examples - when you're finding P(E or F), be sure to look for outcomes that E and F have in common.

The Complement Rule

I think the best way to introduce the last idea in this section is to consider an example. Let's look at a deck of standard playing cards again:



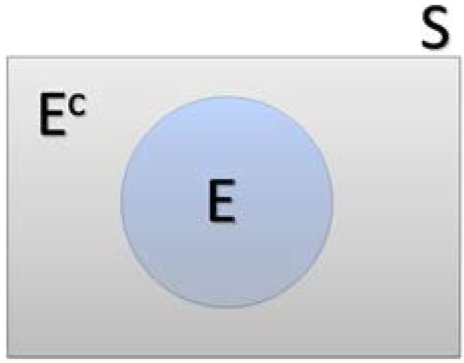
And let's define event E = a card less than a King is drawn. If I ask you to find P(E), you're not going to count them up. (You weren't going to, were you?!) No - you'll say there are 52 cards all together, and there are 4 kings, so therefore there must be 48 cards less than a King. So P(E) = 48/52 = 12/13.

The idea that you're already using there is called the **complement**. (That's complement, with an *e*. Not compliment, as in "My, you look pretty today!")

The **complement of E**, denoted E^c, is all outcomes in the sample space that are not in E.

So essentially, the complement of E is *everything but* the outcomes in E. In fact, some texts actually write it as "not E".

How is the complement helpful? Well, you actually already used the key idea in the example above. Let's look at a Venn diagram.



From Section 5.1, we know that P(S) = 1. Clearly, E and E^c are disjoint, so $P(E \text{ or } E^c) = P(E) + P(E^c)$. Combining those two facts, we get:

The Complement Rule

$$P(E) + P(E^c) = 1$$

Keep this in mind when you're looking at an event that's fairly complicated. Sometimes it's much easier to find the probability of the complement.

Section 5.3: Independence and the Multiplication Rule

Objectives

By the end of this lesson, you will be able to...

1. identify independent events
2. use the Multiplication Rule for independent events
3. compute "at least" probabilities

Independence

One of the most important concepts in probability is that of *independent events*.

Two events E and F are **independent** if the occurrence of event E does not affect the probability of event F.

Let's look at a couple examples.

Example 1

Consider the experiment where two cards are drawn without replacement. (*Without replacement* means one is drawn and then the second is drawn without putting the first one back.) Define events E and F this way:

Source: stock.xchng

E = the first card drawn is a King

F = the second card drawn is a King

Are events E and F independent?

[reveal answer]

Example 2

Consider the experiment in which two fair six-sided dice are rolled, and define events E and F as follows:

E = the first die is a 3

F = the second die is a 3

Are events E and F independent?

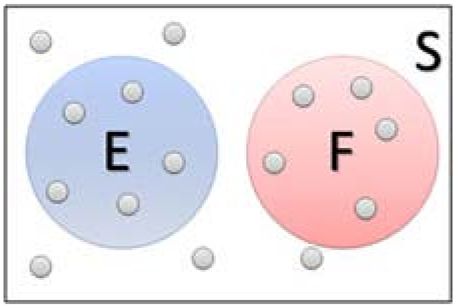
[reveal answer]

Disjoint vs. Independent

It is very common for students to confuse the concepts of *disjoint* (*mutually exclusive*) events with *independent* events. Recall from the last section:

> Two events are **disjoint** if they have no outcomes in common. (Also commonly known as **mutually exclusive** events.)

Here's a Venn diagram of two disjoint events.



Looking at this image, we can see very clearly that if event E occurs (that is, the outcome  is in event E), it cannot possibly be in event F. So E and F are *dependent*, since the occurrence of event E made event F impossible.

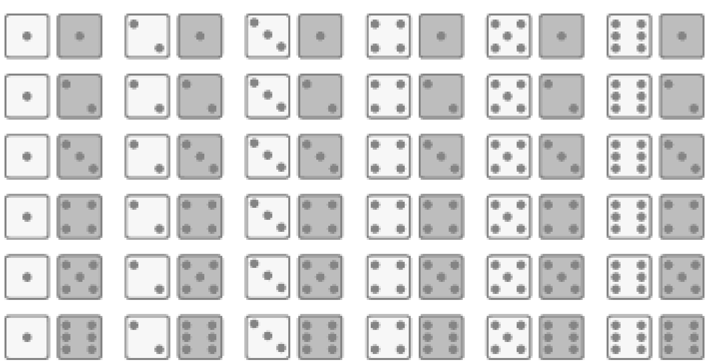
The Multiplication Rule for Independent Events

To introduce the next idea, let's look at the experiment from Example 2, in Section 5.1.

Example 3

The experiment was rolling a fair six-sided die twice. Suppose define the event E:

E = both dice are 2's



The possible outcomes are (1,1), (1,2), (1,3), ... (6,5), and (6,6). Since only one of these is (2,2), we know P(E) = 1/36. Let's look at it another way, though.

$$P(E) = \frac{N(E)}{N(S)} = \frac{1}{36} = \frac{1}{6 \bullet 6} = \frac{1}{6} \bullet \frac{1}{6} = P(E) \bullet P(F)$$

In fact, this will always be true if E and F are independent.

Multiplication Rule for Independent Events

If E and F are independent events, then

$$P(E \text{ and } F) = P(E) \cdot P(F)$$

Example 4

According to data from the American Cancer Society, about 1 in 3 women living in the U.S. will have some form of cancer during their lives.

If three women are randomly selected, what is the probability that they will all contract cancer at some point during their lives?

[reveal answer]

At-least Probabilities

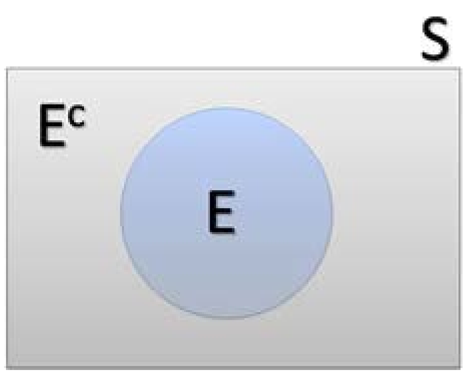
The phrase "at least" can make a seemingly simple problem much more difficult. For example, suppose we're looking at cancer rates in women. And suppose we have a random sample of 5 women. If we're looking for the probability that *at least one* will have some form of cancer, that's really:

P(1 will have cancer) + P(2 will have cancer) + ... + P(all 5 will have cancer)

Instead, it's much easier to use the *Complement Rule*, from Section 5.2.

The Complement Rule

$$P(E) + P(E^c) = 1$$



In our example, the complement of *at least one will have cancer* is *none will have cancer*. So P(at least one will have cancer) = 1 - P(none will have it) = $1 - (1/3)^5 \approx 0.9959$.

Much easier! Keep this idea of *at least* probabilities and the Complement Rule in mind when you're looking at cases like this.

Section 5.4: Conditional Probability and the General Multiplication Rule

Objectives

By the end of this lesson, you will be able to...

1. compute conditional probabilities
2. use the General Multiplication Rule
3. determine the independence of events

Conditional Probability

Remember in Example 3, in Section 5.3, about rolling two dice? In that example, we said that events E (the first die is a 3) and F (the second die is a 3) were *independent*, because the occurrence of E didn't effect the probability of F. Well, that won't always be the case, which leads us to another type of probability called *conditional probability*.

Conditional Probability

The notation P(F|E) is read "the probability of F given E" and represent the probability that event F occurs, given that event E has already occurred.

Let's look again at Example 1 from that same section.

Example 1

Consider the experiment where two cards are drawn without replacement. (*Without replacement* means one is drawn and then the second is drawn without putting the first one back.) Define events E and F this way:

E = the first card drawn is a King

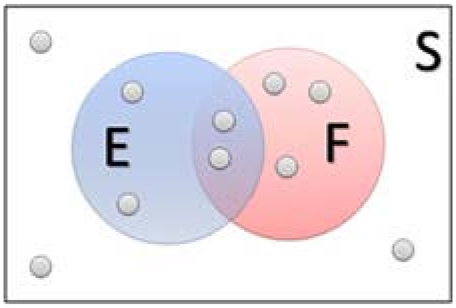
F = the second card drawn is a King

Find P(F|E).

[reveal answer]

It can be helpful again to look at a Venn diagram to illustrate the idea. Let's look at this one that we used back in section 5.2.

Example 2



Find P(E|F).

[reveal answer]

One more example.

Example 3

Let's consider a survey given to 52 students in a Basic Algebra course at ECC, with the following responses to the statement "I enjoy math."

	Strongly Agree	Agree	Neutral	Disagree	Strongly Disagree	Total
Male	6	10	3	0	0	19
Female	8	14	7	4	0	33
Total	14	24	10	4	0	52

What is the probability that a student enjoys math (*Agree* or *Strongly Agree*) given that the student is a female?

[reveal answer]

The Monty Hall Problem

An interesting example of conditional probability is the classic **Monty Hall Problem**. This is based on an old game show, where the host would show three doors. Behind one was a new car, and behind the others were goats. The twist was, once you made your choice, Monty would open one of the other doors showing a goat. The question then - **should you switch?** The answer is different from what you would think. Here's another video from Clive Rix at the University of Leicester in Leicester, England:

[redacted]

Don't believe it? Try this interactive feature from the New York Times, or watch this video from the show Numb3rs.

Wow - you never know where conditional probability can be applied!

The General Multiplication Rule

Let's look again at the experiment from Example 1 in Section 5.3.

Example 4

Source: stock.xchng

Consider the experiment where two cards are drawn without replacement. (*Without replacement* means one is drawn and then the second is drawn without putting the first one back.) Define events E and F this way:

E = the first card drawn is a King

F = the second card drawn is a King

How would we find P(E and F)?

We know from Example 1 that E and F are not independent, so we know we can't use the Multiplication Rule for Independent Events. It's probably not too difficult to see how we might do it, though.

$$\begin{aligned} P(E \text{ and } F) &= P(\text{first is King and second is King}) \\ &= P(\text{first is King}) \bullet P(\text{second is King given first is King}) \\ &= (4/52)(3/51) \\ &\approx 0.0045 \end{aligned}$$

Or in other words, P(E and F) = P(E) • P(F|E)

This idea is actually a version of the *Multiplication Rule for Independent Events*, and is called the *General Multiplication Rule*.

General Multiplication Rule

The probability that two events E and F both occur is

$$P(E \text{ and } F) = P(E) \bullet P(F|E)$$

Example 5

Source: stock.xchng

Let's try a new probability experiment. This time, consider a bag of marbles, containing 10 red, 20 blue, and 15 green marbles. Suppose that two marbles are drawn without replacement. (The first marble is not put back in the bag before drawing the second.)

What is the probability that *both* marbles drawn are red?

[reveal answer]

Checking for Independence

If you recall, in Section 5.3, we defined what it meant for two events to be independent:

Two events E and F are **independent** if the occurrence of event E does not affect the probability of event F.

Looking at this in terms of conditional probability, if the occurrence of E doesn't affect the probability of F, then P(F|E) = P(F). This is a good way to test for independence. In fact, we can redefine independence using this concept.

Two events E and F are **independent** if P(F|E) = P(F).

Let's use this new definition in an example to determine if two events are independent.

Example 6

Let's again use the data from Example 3 and the survey given to 52 students in a Basic Algebra course at ECC, with the following responses to the statement "I enjoy math."

	Strongly Agree	Agree	Neutral	Disagree	Strongly Disagree	Total
Male	6	10	3	0	0	19
Female	8	14	7	4	0	33
Total	14	24	10	4	0	52

Suppose a student is selected at random from those surveyed and we define the events E and F as follows:

E = student selected is female

F = student enjoys math

Are events E and F independent?

[reveal answer]

Section 5.5: Counting Techniques

Objectives

By the end of this lesson, you will be able to...

1. solve counting problems using the Multiplication Rule
2. solve counting problems using permutations
3. solve counting problems using combinations
4. solve counting problems involving permutations with non-distinct items
5. compute probabilities involving permutations and combinations

Do you remember the classical method for calculating probabilities from Section 5.1?

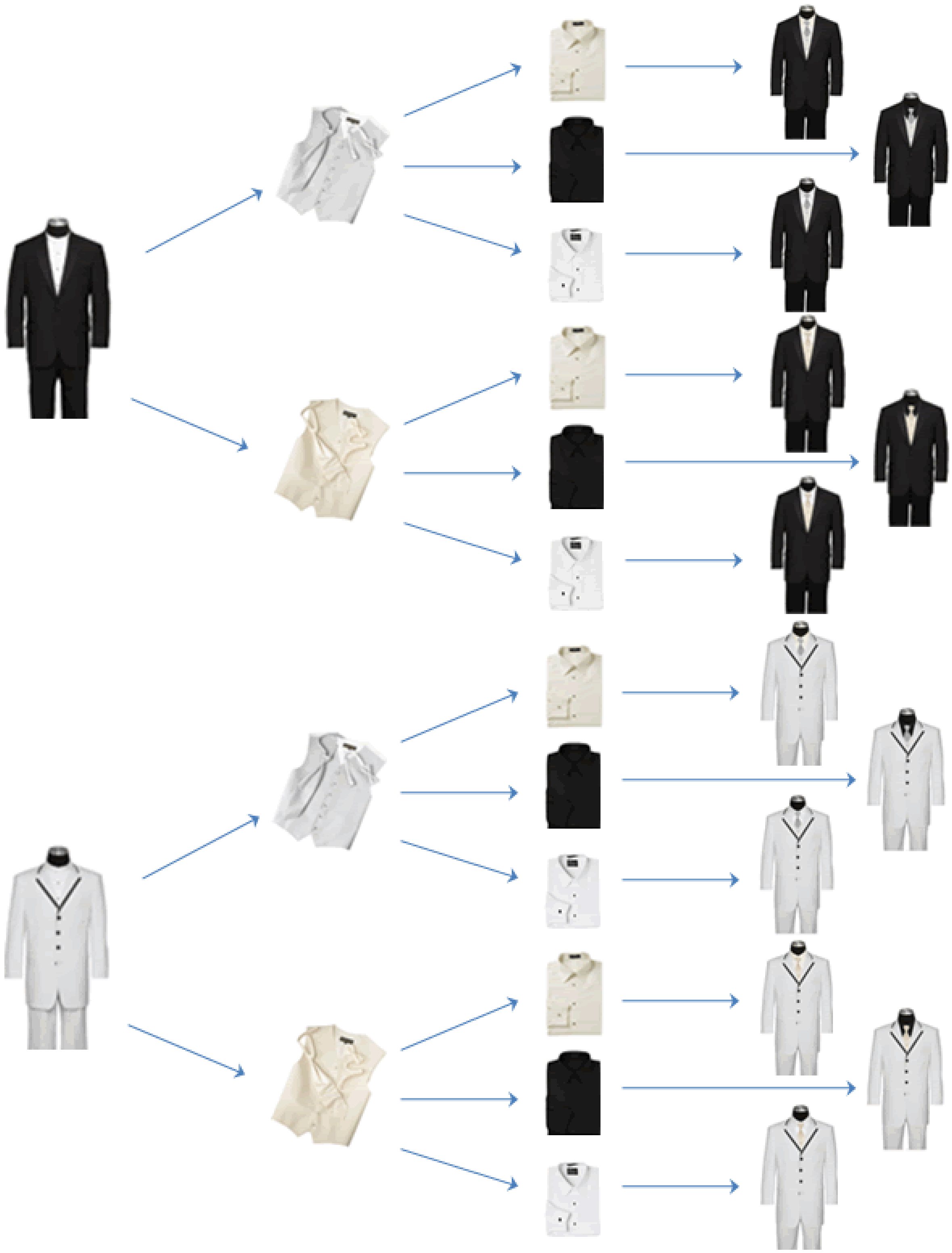
$$P(E) = \frac{\text{number of ways E can occur}}{\text{total number of possible outcomes}} = \frac{N(E)}{N(S)}$$

Well, sometimes counting the "number of ways E can occur" or the "total number of possible outcomes" can be fairly complicated. In this section, we'll learn several counting techniques, which will help us calculate some of the more complicated probabilities.

The Multiplication Rule of Counting

Let's suppose you're preparing for a wedding, and you need to pick out tuxedos for the groomsmen. Men's Tuxedo Warehouse has a Build-A-Tux feature which allows you to look at certain combinations and build your tuxedo online. Let's suppose you have the components narrowed down to two jackets, two vest and tie combinations, and three shirt colors. How many total combinations might there be?

A good way to help understand this type of situation is something called a **tree diagram**. We begin with the jacket choices, and then each jacket "branches" out into the two vest and tie combinations, and then each of those then "branches" out into the three shirt combinations. It might look something like this:



In total, it looks like we have 12 possible combinations of jackets, vests, and shirts. (Of course, some may not fit your fashion sense, but that's another question all-together...)

Isn't there an easier way to do this? Why yes, there is! Think of it this way, for *each* jacket choice, there are two vest and tie choices. That gives us 4 total jacket and vest/tie combinations. Then, for *each of those*, there are three shirt choices, giving us a total of 12.

In general, we *multiply* the number of ways to make each choice, so...

total number of outfits = (number of jackets)•(number of vest/ties)•(number of shirts)

That leads us to the Multiplication Rule of Counting:

Multiplication Rule of Counting

If a task consists of a sequence of choices in which there are p ways to make the first choice, q ways to make the second, etc., then the task can be done in

$$p \cdot q \cdot r \cdot \ldots$$

different ways.

Let's try some examples.

Example 1

How many 7-character license plates are possible if the first three characters must be letters, the last four must be digits 0-9, and repeated characters are allowed?

[reveal answer]

Example 2

Many garage doors have remote-access keypads outside the door. Let's suppose a thief approaches a particular garage and notices that four particular numbers are well-used. If we assume the code uses all four numbers exactly once, how many 4-digit codes does the thief have to try?

Source: Sears

[reveal answer]

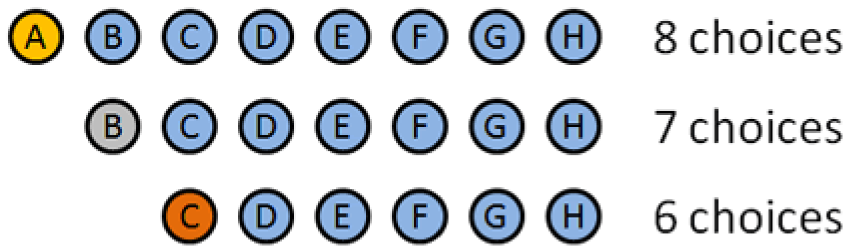
Example 2, from earlier this section is an example of particular counting technique called a **permutation**. Rather than giving you formulas and examples myself, I'd like to make another reference to some content from one of my favorite web sites, BetterExplained. Here's what the author, Kalid Azad writes about permutations:

Permutations: The hairy details

Let's start with permutations, or **all possible ways** of doing something. We're using the fancy-pants term "permutation", so we're going to care about every last detail, including the order of items. Let's say we have 8 people:

1. Alice
2. Bob
3. Charlie
4. David
5. Eve
6. Frank
7. George
8. Horatio

How many ways can we pick a Gold, Silver, and Bronze medal for "Best friend in the world"?



We're going to use permutations since the order we hand out these medals matter. Here's how it breaks down:

Gold medal: 8 choices: A B C D E F G H (Clever how I made the names match up with letters, eh?). Let's say A wins the Gold.
Silver medal: 7 choices: B C D E F G H. Let's say B wins the silver.
Bronze medal: 6 choices: C D E F G H. Let's say… C wins the bronze.

We picked certain people to win, but the details don't matter: we had 8 choices at first, then 7, then 6. The total number of options was 8 * 7 * 6 = 336.

Let's look at the details. We had to order 3 people out of 8. To do this, we started with all options (8) then took them away one at a time (7, then 6) until we ran out of medals.

We know the factorial is: $8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

Unfortunately, that does too much! We only want 8 * 7 * 6. How can we "stop" the factorial at 5?

This is where permutations get cool: notice how we want to get rid of 5*4*3*2*1. What's another name for this? 5 factorial!

So, if we do 8!/5! we get:

$$\frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 8 \cdot 7 \cdot 6$$

And why did we use the number 5? Because it was left over after we picked 3 medals from 8. So, a better way to write this would be:

$$\frac{8!}{(8-3)!}$$

where 8!/(8-3)! is just a fancy way of saying "Use the first 3 numbers of 8!". If we have **n** items total and want to pick **k** in a certain order, we get:

$\frac{n!}{(n-k)!}$ just means "Use the first k numbers of n!"

And this is the fancy permutation formula: You have **n** items and want to find the number of ways **k** items can be ordered:

$$P(n,k) = \frac{n!}{(n-k)!}$$

Source: BetterExplained, Kalid Azad
Article: Easy Permutations and Combinations
Used with permission.

As a side note, your text uses the notation ${}_nP_k$ rather than Kalid's P(n,k). I've seen both used, though the later tends to be more prevalent in higher-level math classes. We'll stick with the textbook version, just to be consistent.

Permutations of *n* Distinct Objects Taken *r* at a Time

The number of arrangements of *r* objects chosen from *n* objects in which

1. the *n* objects are distinct,
2. repeats are not allowed,
3. order matters,

is given by the formula ${}_nP_k = \frac{n!}{(n-r)!}$.

OK, let's try a couple.

Example 3

Suppose an organization elects its officers from a board of trustees. If there are 30 trustees, how many possible ways could the board elect a president, vice-president, secretary, and treasurer?

[reveal answer]

Example 4

Suppose you're given a list of 100 desserts and asked to rank your top 3. How many possible "top 3" lists are there?

[reveal answer]

In the previous page, we talked about the number of ways to choose *k* objects from *n* if the ordered mattered - like giving medals, electing officers, or pickling favorite desserts. What if order doesn't matter, like picking members of a committee?

Again, I'll let Kalid Azad explain.

Combinations, Ho!

Combinations are easy going. Order doesn't matter. You can mix it up and it looks the same. Let's say I'm a cheapskate and can't afford separate Gold, Silver and Bronze medals. In fact, I can only afford empty tin cans.

How many ways can I give 3 tin cans to 8 people?

Well, in this case, the order we pick people doesn't matter. If I give a can to Alice, Bob and then Charlie, it's the same as giving to Charlie, Alice and then Bob. Either way, they're going to be equally disappointed.

This raises and interesting point — we've got some redundancies here. Alice Bob Charlie = Charlie Bob Alice. For a moment, let's just figure out how many ways we can rearrange 3 people.

Well, we have 3 choices for the first person, 2 for the second, and only 1 for the last. So we have 3 * 2 * 1 ways to re-arrange 3 people.

Wait a minute… this is looking a bit like a permutation! You tricked me!

Indeed I did. If you have N people and you want to know how many arrangements there are for **all** of them, it's just N factorial or N!

So, if we have 3 tin cans to give away, there are 3! or 6 variations for every choice we pick. If we want to figure out how many combinations we have, we just **create all the permutations and divide by all the redundancies**. In our case, we get 336 permutations (from above), and we divide by the 6 redundancies for each permutation and get 336/6 = 56.

The general formula is

$$C(n,k) = \frac{P(n,k)}{k!}$$

which means "Find all the ways to pick k people from n, and divide by the k! variants". Writing this out, we get our **combination formula**, or the number of ways to combine k items from a set of n:

$$C(n,k) = \frac{n!}{(n-k)!k!}$$

Source: BetterExplained, Kalid Azad
Article: Easy Permutations and Combinations
Used with permission.

As a side note, your text uses the notation ${}_nC_k$ rather than Kalid's C(n,k). As with permutations, we'll stick with the textbook version, just to be consistent.

Combinations of *n* Distinct Objects Taken *r* at a Time

The number of arrangements of *n* objects using $r \le n$ of them, in which

1. the *n* objects are distinct,
2. repeats are not allowed,
3. order does not matter,

is given by the formula ${}_nC_k = \dfrac{n!}{r!\,(n-r)!}$.

All right, let's try this new one out.

Example 5

Let's consider again the board of trustees with 30 members. In how many ways could the board elect four members for the finance committee?

[reveal answer]

Example 6

Suppose you're a volleyball tournament organizer. There are 10 teams signed up for the tournament, and it seems like a good idea for each team to play every other team in a "round robin" setting, before advancing to the playoffs. How many games are possible if each team plays every other team once?

Source: stock.xchng

[reveal answer]

This second type is less common. What if we want to know how many ways to order *n* objects, but they're not all distinct? Here's an example to illustrate:

Example 7

In how many ways could the letters in the word STATISTICS be rearranged?

The answer is a little tricky. Think of the rearranged words as places for letters to go. Something like this:

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

In STATISTICS, we have the following letters:

3 S's
3 T's
2 I's
1 A
1 C

We can't really say that there are 4 choices for the first letter and proceed from there, since the number of choices for the second letter depend on which letter was chosen for the first.

Instead, we *choose the spots* for each of the letters. First, pick 3 of the 10 spots for the S's. We can do that in ${}_{10}C_3$ ways. Then pick 3 spots for the 3 T's. We can do that in ${}_7C_3$ ways. Similarly, we can pick the spots for the I's, the A, and the C in ${}_4C_2$, ${}_2C_1$, and ${}_1C_1$ ways, respectively. In total, that means we rearrange the letters in:

${}_{10}C_3 \cdot {}_7C_3 \cdot {}_4C_2 \cdot {}_2C_1 \cdot {}_1C_1$ ways

It may just be me, that is *really* messy. Oddly enough, writing out the combinations reveals a nice way to simplify it:

$$\frac{10!}{7! \cdot 3!} \cdot \frac{7!}{4! \cdot 3!} \cdot \frac{4!}{2! \cdot 2!} \cdot \frac{2!}{1! \cdot 1!} \cdot \frac{1!}{1! \cdot 0!} = \frac{10!}{3! \cdot 3! \cdot 2! \cdot 1! \cdot 1!}$$

Permutations with Non-distinct Items

The number of permutations of *n* objects, where there are n1 of the 1st type, n2 of the 2nd type, etc, is

$$\frac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_k!}$$

One quick example:

Example 8

In how many ways can the word REARRANGE be rearranged?

[reveal answer]

Section 5.6: Putting It Together: Which Method Do I Use?

Objectives

By the end of this lesson, you will be able to...

1. determine the appropriate probability rule to use
2. determine the appropriate counting technique to use

The last thing we really need to do is connect all these counting techniques to the stated purpose - probability. We'll start by first focusing on choosing the correct probability rule. Once we have that down, we'll continue to the next stage of selecting the correct counting technique.

Probability Rules

Let's start by reviewing the probability rules from the previous sections.

The Basic Principle of Probability (Classical Method)

$$P(E) = \frac{\text{number of ways E can occur}}{\text{total number of possible outcomes}} = \frac{N(E)}{N(S)}$$

The General Addition Rule

$$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$$

The Complement Rule

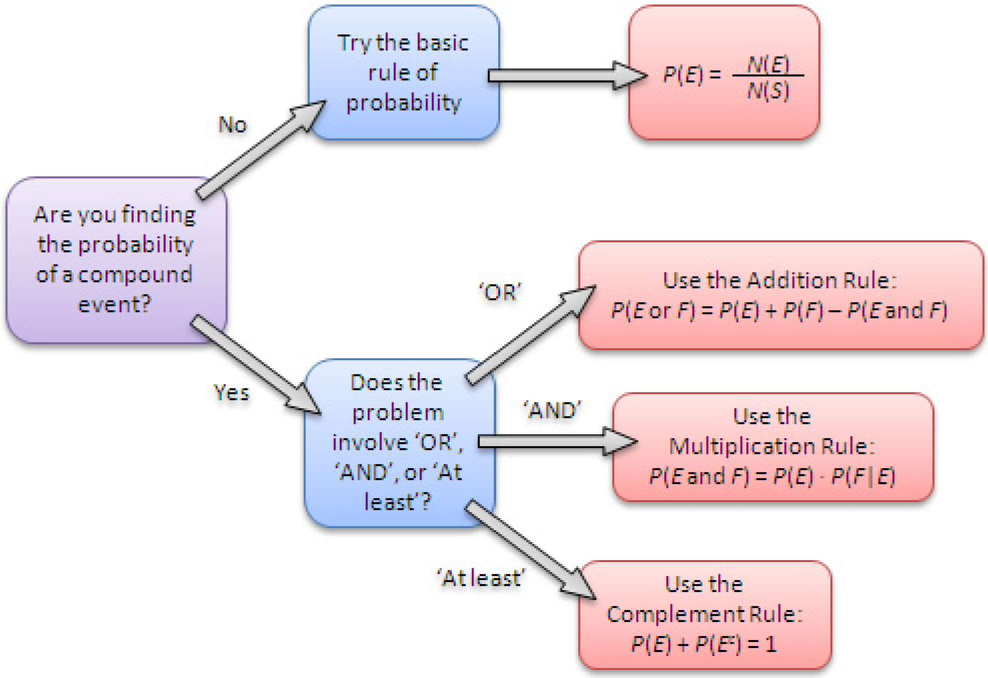
$$P(E) + P(E^c) = 1$$

General Multiplication Rule

The probability that two events E and F both occur is

$$P(E \text{ and } F) = P(E) \cdot P(F|E)$$

Our task then, is choosing the correct rule. Here's a simplified version of the decision-making flowchart from your textbook.



Let's try an example together.

Example 1

Problem: Suppose a basketball player makes 80% of her free throws. If she shoots five free throws, find the probability that she makes at least one basket.

Solution: Let's try following our flowchart:

- Are you finding the probability of a compound event?
  Yes.
- Does the problem involve 'AND', 'OR', or 'At least'?
  At least.
- Use the Complement Rule.

The Complement Rule is $P(E) + P(E^C) = 1$. In our case, we have something like this:

P(she makes at least one) + P(she misses all five) = 1

We can find what we want by solving for it:

P(she makes at least one) = 1 - P(she misses all five)

Now, we need to look at the probability that she misses all five. Let's try the sequence again:

- Are you finding the probability of a compound event?
  Yes.
- Does the problem involve 'AND', 'OR', or 'At least'?
  AND. (We want the probability that she misses the first **and** the second **and** the third, etc)
- Use the Multiplication Rule.

If we assume the individual attempts are independent (granted, a big assumption), we can find the probability that she misses all five by multiplying the probability that she misses one by itself five times.

P(she makes at least one)
= 1 - P(she misses all five)
= 1 - (0.2)(0.2)(0.2)(0.2)(0.2)
=0.99968

So she'll make at least one basket 99.97% of the time!

All right, it's time for you to try some yourself. Here we go...

Example 2 Consider a standard 52-card deck of playing cards. A single card is drawn at random, with the following events defined:

A = a diamond is drawn
B = a face card is drawn (*face cards* are Jacks, Queens, or Kings)

Find P(A or B).

[reveal answer]

Example 3

Suppose a fair die is tossed and a fair coin is flipped. Find the probability that the die is even and the coin is heads.

[reveal answer]

Example 4

Texas Hold'Em is form of poker regaining popularity recently due to the exposure of the World Series of Poker on ESPN. The game is fairly complex, but components of it can be relatively easy to understand.

Source: stock.xchng

From Wikipedia
Hold 'em is a community card game where each player may use any combination of the five community cards and the player's own two hole cards to make a poker hand, in contrast to poker variants like stud or draw where each player holds a separate individual hand.

The game is played by first dealing every player two cards. Then three card are dealt face-up in the middle of the table (this is called *the flop*). All players are allowed to use these "community" cards in the middle of the table along with the two best . Later, two more card are dealt, with a total of five "community" cards for all players to use in their hands.

Suppose you are dealt A♥Q♥. If we assume the other 50 cards are all equally likely to be dealt during the flop, what is the probability that the next three cards are also hearts, giving you a flush (five of the same suit) after the flop?

Just focus on choosing the correct probability rule - don't worry about actually finding the probability.

[reveal answer]

Of course, this is only a small selection of the wide variety of problems possible. Be sure to complete all of the MyMathLab homework and the practice exam questions. You might also consider looking at some of the unassigned odd problems in your text for extra practice, or additional practice through MyMathLab.

Once you've determined which probability rule to apply, you often need some counting techniques in order to complete the problem. For many students, this aspect of probability questions is the most troublesome. Like the previous page (and your text), we'll illustrate the main ideas with a flow chart.

The Counting Techniques

As with the probability rules, we need to first review some of our different counting techniques, beginning with the most important.

Multiplication Rule of Counting

If a task consists of a sequence of choices in which there are *p* ways to make the first choice, *q* ways to make the second, etc., then the task can be done in

$$p \bullet q \bullet r \bullet \ldots$$

different ways.

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is given by the formula ${}_nP_k = \dfrac{n!}{(n-r)!}$.

Combinations of *n* Distinct Objects Taken *r* at a Time

The number of arrangements of *n* objects using $r \leq n$ of them, in which

1. the *n* objects are distinct,
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Permutations with Non-distinct Items

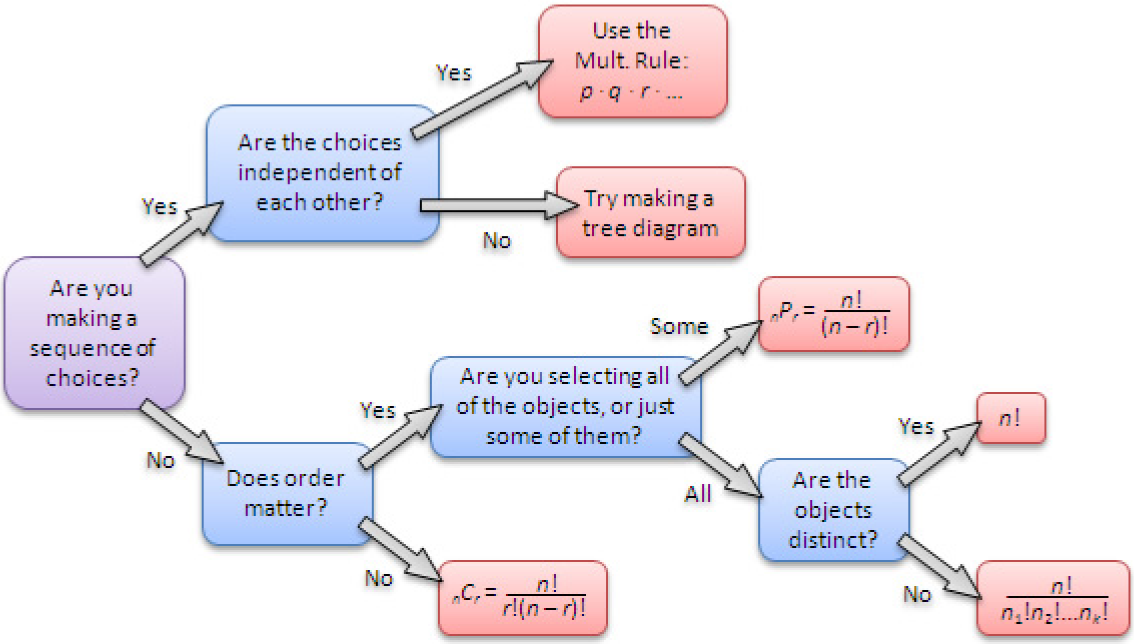
The number of permutations of *n* objects, where there are n1 of the 1st type, n2 of the 2nd type, etc, is

$$\frac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_k!}$$

Choosing the Appropriate Counting Technique

The trouble, then, becomes choosing which one of these (or more than one) applies to a particular problem.

As we did with the probability rules, we'll illustrate this with a decision-making flowchart. In this case, it's the same one that's in your text.



Let's try one example together.

Example 5

Questions: In the Powerball lottery, there are 5 balls numbered 1-55, and an additional "powerball" numbered 1-42. To win the grand prize, you must match all 5 and the powerball. **What is the probability of winning the grand prize with one ticket?**

Solutions: To answer this question, we need to remember the basic probability rule. Expanded for this example, it would be something like this:

$$P(\text{winning}) = \frac{\text{number of ways to win}}{\text{total number of possible outcomes}}$$

The number of ways to win is easy - there's only one!

For the total number of possible outcomes, let's consider our flowchart.

- **Are we making a sequence of choices?**
  Yes. We're first counting the number of ways for the 5 balls to be drawn, and then we need to count the number of ways for the powerball to be drawn.
- **Are the choices independent of each other?** (In other words, is the powerball independent of what happens in the first 5 balls?) **Yes.**
- **Use the Multiplication Rule of Counting.**

So we need to multiply the number of ways to do each step.

total # of outcomes
= (# of ways for the 5 to be drawn)•(# of ways for powerball)

Since order doesn't matter for the 5 balls, that part is a combination of 5 from 55. The powerball is simply 42, since there are 42 choices.

The total number of outcomes is thus ${}_{55}C_5 \cdot 42 = 146{,}107{,}962$.

The probability is then $1/146{,}107{,}962 \approx 0.000\,000\,007$

That means a $1 ticket has only about a 1 in 150 million chance of winning the grand prize!

Here are some examples to try. Some are counting questions and some are actual probability questions, but the probability rule shouldn't be the hard part. (Hint: They can all use the classical method!)

Example 6

Suppose you are told to create a password that has the following rules:

1. it must consist of exactly 4 characters,
2. one must be a number,
3. one must be a capitalized, and
4. the letters must all differ.

How many possible passwords are there following these rules?

[reveal answer]

Example 7

Suppose you have a bag of 20 blue marbles and 40 red marbles. What is the probability that if 5 are drawn without replacement, 2 are blue and 3 are red?

[reveal answer]

Example 8

Source: stock.xchng

In Example 4, you worked on choosing the correct probability rule for a Texas Hold'Em situation. Use your rule to actually calculate the probability. Here's the problem again:

Texas Hold'Em is form of poker regaining popularity recently due to the exposure of the World Series of Poker on ESPN. The game is fairly complex, but components of it can be relatively easy to understand.

> **From Wikipedia**
> Hold 'em is a community card game where each player may use any combination of the five community cards and the player's own two hole cards to make a poker hand, in contrast to poker variants like stud or draw where each player holds a separate individual hand.

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[reveal answer]

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