Bounded Monotonic Sequence

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.

Convergence of a Geometric Sequence

A geometric series with ratio r diverges if $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \qquad 0 < |r| < 1$$

*n*th Term

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Integral Test

If f is positive, continuous, and decreasing for all $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)dx$$

either both converge or both diverge. (*Note*: These conditions need only be satisfied for all $x \geq N > 1$.)

p-Series

The *p*-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

1. converges if $p > 1$, and
2. diverges if $0 < p \leq 1$.

Direct Comparison Test

Let $0 < a_n \leq b_n$ for all n.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Limit Comparison Test

Suppose that $a_n > 0$, $b_n > 0$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

where L is *finite and positive*. Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Alternating Series Test

Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converge if the following two conditions are met.

1. $\lim_{n\to\infty} a_n = 0$
2. $a_{n+1} \le a_n$, for all n*

* This can be modified to require only that $a_{n+1} \le a_n$ for all n greater than some integer N.

Alternating Series Remainder

$$|S - S_N| = |R_N| \le a_{N+1}$$

Absolute Convergence

If the series $\sum |a_n|$ converges, then the series $\sum a_n$ also converges.

Ratio Test

Let $\sum a_n$ be a series of non-zero terms.

1. $\sum a_n$ converges absolutely if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$.
2. $\sum a_n$ diverges if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ or $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$.
3. The Ratio Test is inconclusive if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.

Root Test

Let $\sum a_n$ be a series.

1. $\sum a_n$ converges absolutely if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.
2. $\sum a_n$ diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$.
3. The Root Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

Taylor Polynomial

If *f* has *n* derivatives at *c*, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the ***n*th Taylor polynomial for *f* at c**. If $c = 0$, then

$$P_n(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \cdots + \frac{f^{(n)}(0)}{n!}(x)^n$$

is also called the ***n*th Maclaurin polynomial for *f***.

Power Series

If *x* is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a **power series**. More generally, and infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots$$

is called a **power series centered at *c***, where *c* is a constant.

Convergence of a Power Series

For a power series centered at *c*, precisely one of the following is true.

1. The series converges only at *c*.
2. There exists a real number $R > 0$ such that the series converges absolutely for $|x-c| < R$, and diverges for $|x-c| > R$.
3. The series converges absolutely for all *x*.

The number *R* is the **radius of convergence** of the power series. If the series converges only at *c*, $R = 0$, and if the series converges for all *x*, $R = \infty$. The set of values of *x* for which the power series converges is the **interval of convergence** of the power series.

The Form of a Convergent Power Series

If *f* is represented by a power series $f(x) = \sum a_n (x-c)^n$ for all *x* in an open interval *I* containing *c*, then $a_n = f^{(n)}(c)/n!$ and

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

Taylor Series

If a function *f* has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

is called the **Taylor series for $f(x)$ at *c***. Moreover, if $c = 0$, then the series is the **Maclaurin series for *f***.

<u>Guideline for Finding a Taylor Series</u>

1. Differentiate $f(x)$ several times and evaluate each derivative at c.

   $$f(c),\ f'(c),\ f''(c),\ f'''(c),\ \ldots,\ f^{(n)}(c),\ \ldots$$

   Try to recognize a pattern in these numbers.

2. Use the sequence developed in the first step to form the Taylor coefficients $a_n = f^{(n)}(c)/n!$, and determine the interval of convergence for the resulting power series

   $$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

3. Within this interval of convergence, determine whether or not the series converges to $f(x)$.

Power Series for Elementary Functions

Function	*Interval of Convergence*
$\frac{1}{x} = 1-(x-1)+(x-1)^2-(x-1)^3+(x-1)^4-\cdots(-1)^n(x-1)^n+\cdots$	$0<x<2$
$\frac{1}{1+x} = 1-x+x^2-x^3+x^4-\cdots+(-1)^n x^n+\cdots$	$-1<x<1$
$\ln x = (x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots+\frac{(-1)^n(x-1)^n}{n}+\cdots$	$0<x\le 2$
$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\frac{x^n}{n!}+\cdots$	$-\infty<x<\infty$
$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\cdots$	$-\infty<x<\infty$
$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots+\frac{(-1)^n x^{2n}}{(2n)!}+\cdots$	$-\infty<x<\infty$
$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\cdots+\frac{(-1)^n x^{2n+1}}{2n+1}+\cdots$	$-1\le x\le 1$
$\arcsin x = x+\frac{x^3}{2\cdot 3}+\frac{1\cdot 3x^5}{2\cdot 4\cdot 5}-\frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7}+\cdots+\frac{(2n)!\,x^{2n+1}}{(2^n n!)^2(2n+1)}+\cdots$	$-1\le x\le 1$
$(1+x)^k = 1+kx+\frac{k(k-1)x^2}{2!}+\frac{k(k-1)(k-2)x^3}{3!}+\frac{k(k-1)(k-2)(k-3)x^4}{4!}+\cdots$	$-1<x<1$*

* *The convergence at* $x=\pm 1$ *depends on the value of k.*