Proof of the Integral Test

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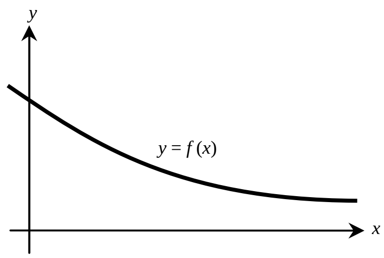
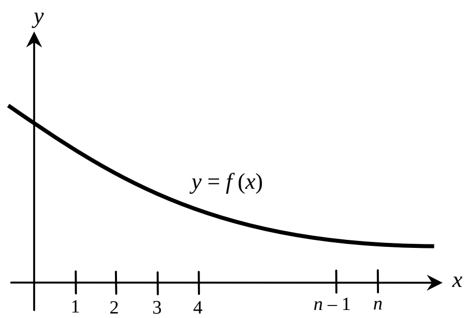
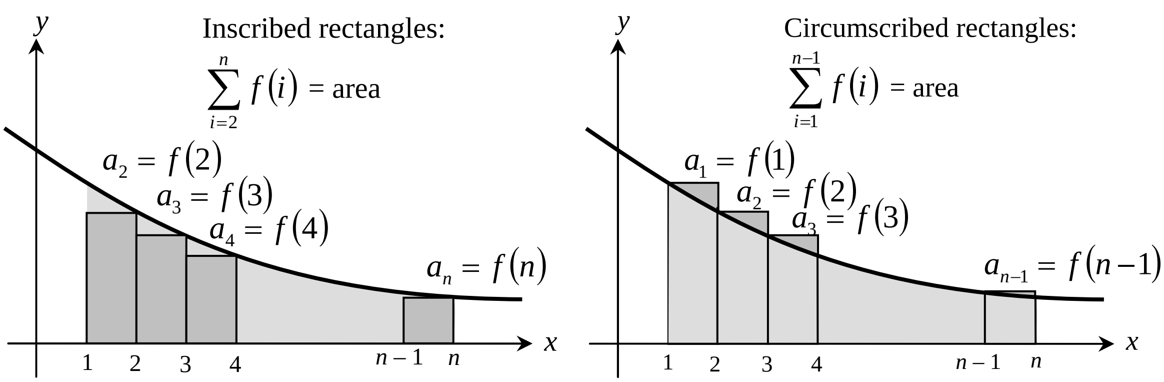
f positive, continuous, and decreasing for $x \geq 1$ means f has the general shape:



Partition the interval $[1, n]$ into $n-1$ unit intervals.



Next, consider $n-1$ inscribed and circumscribed rectangles as illustrated below:



From the two sets of rectangles, we can see that

$$\sum_{i=2}^{n} f(i) \leq \int_{1}^{n} f(x)dx \leq \sum_{i=1}^{n-1} f(i) \qquad (1)$$

Using the nth partial sum, $S_n = f(1) + f(2) + \cdots + f(n)$, statement (1) can be written as:

$$S_n - f(1) \le \int_1^n f(x)dx \le S_{n-1} \qquad (2)$$

If we assume that $\int_1^\infty f(x)dx$ converges to L, it follows from statement (2) that for $n \ge 1$

$$S_n - f(1) \le L \quad \Rightarrow \quad S_n \le L + f(1) \qquad (3)$$

Since f is positive for all x, a_n is positive for all n and S_n is monotonic increasing. Statement (3) shows that S_n is also bounded above, so by Theorem 9.5, the sequence $\{S_n\}$ must converge. Therefore, the series $\sum a_n$ converges.

To show the other direction, we go back to statement (2). It follows that for $n \ge 1$

$$S_{n-1} \ge \int_1^n f(x)dx \qquad (4)$$

If we assume $\int_1^\infty f(x)dx$ diverges, then statement (4) implies that the sequence $\{S_n\}$ also diverges. So, the series $\sum a_n$ diverges.