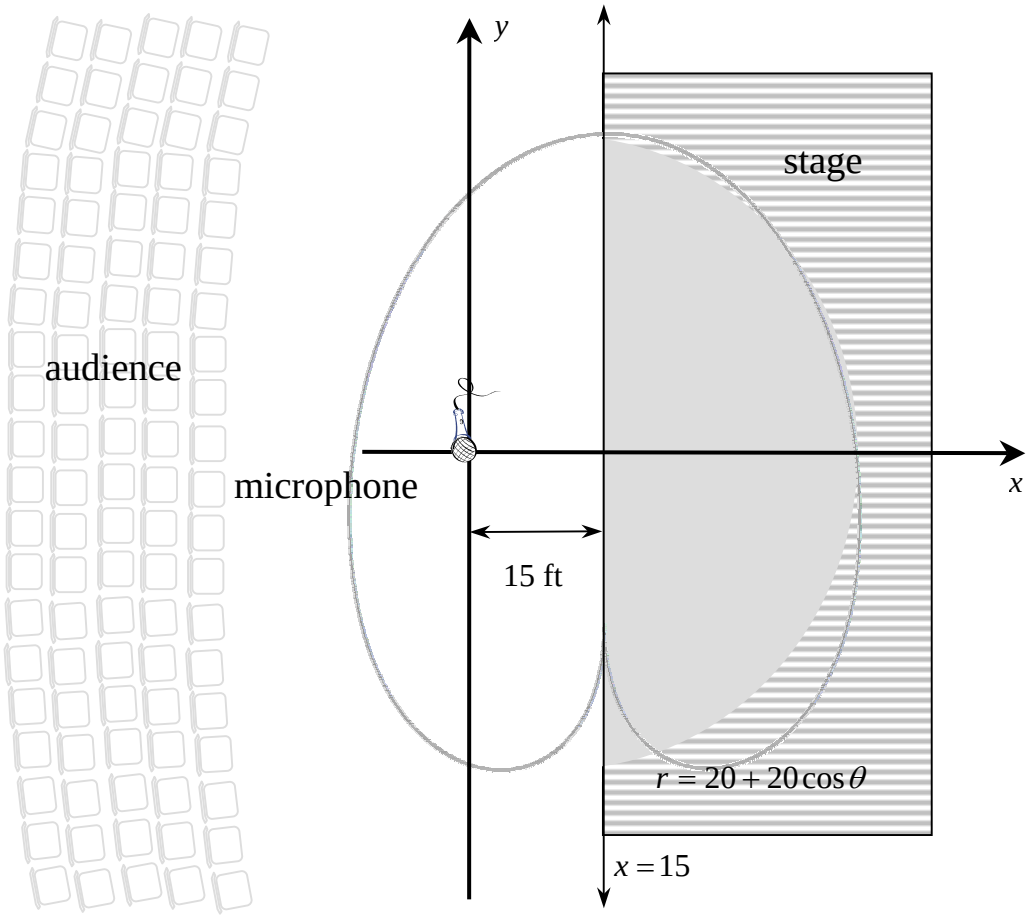
Microphones with a Cardioid Pickup Pattern

A microphone with a cardioid pickup pattern is often the choice of sound engineers recording live performances. The cardioid pattern offers good frontal sensitivity, while suppressing audience noise in back of the microphone. Consider the figure below. Find the area on the stage that lies within the optimal pickup range of the microphone, if the boundaries of this region are given by the cardioid $r = 20 + 20\cos\theta$ and the vertical line $x = 15$ (the microphone has been placed 15 feet from the edge of the stage).



Solution:

First, rewrite $x = 15$ in polar coordinates.

$$\begin{aligned} x &= 15 \\ r\cos\theta &= 15 \\ r &= \frac{15}{\cos\theta} \\ r &= 15\sec\theta \end{aligned}$$

Next, find the θ values of the points of intersection.

$$\begin{aligned} 20 + 20\cos\theta &= \frac{15}{\cos\theta} \\ 20\cos\theta + 20\cos^2\theta &= 15 \\ 4\cos^2\theta + 4\cos\theta - 3 &= 0 \\ (2\cos\theta + 3)(2\cos\theta - 1) &= 0 \\ \cos\theta = -\frac{3}{2} \quad &\text{or} \quad \cos\theta = \frac{1}{2} \\ \theta &= \pm\frac{\pi}{3} \end{aligned}$$

The integral for the area of the region is then:

$$\begin{aligned} &\frac{1}{2}\int_{-\pi/3}^{\pi/3}\left[(20 + 20\cos\theta)^2 - (15\sec\theta)^2\right]d\theta \\ &= 2\cdot\frac{1}{2}\int_0^{\pi/3}\left[400 + 800\cos\theta + 400\cos^2\theta - 225\sec^2\theta\right]d\theta \\ &= \int_0^{\pi/3}\left[400 + 800\cos\theta + 400\frac{1+\cos 2\theta}{2} - 225\sec^2\theta\right]d\theta \\ &= \int_0^{\pi/3}\left[600 + 800\cos\theta + 200\cos 2\theta - 225\sec^2\theta\right]d\theta \\ &= \left(600\theta + 800\sin\theta + 100\sin 2\theta + 225\tan\theta\right)\Big|_0^{\pi/3} \\ &= 600\left(\frac{\pi}{3}\right) + 800\left(\frac{\sqrt{3}}{2}\right) + 100\left(\frac{\sqrt{3}}{2}\right) - 225\sqrt{3} - 0 \\ &= 200\pi + 225\sqrt{3} \\ &\approx 1018\ ft^2 \end{aligned}$$