Mth133 – Calculus – **Practice Exam Solutions**

1.

a. 3
b. 5
c. 3
d. does not exist
e. No, because $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$.
f. It is not continuous, because $f(0)$ does not exist.

2.

a. $\lim_{x\to 3}\frac{x^2-x-6}{x-3}=\lim_{x\to 3}\frac{(x-3)(x+2)}{x-3}=\lim_{x\to 3}(x+2)=3+2=5$

b. $\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}=\lim_{x\to 3}\frac{\left(\sqrt{x+1}-2\right)}{(x-3)}\cdot\frac{\left(\sqrt{x+1}+2\right)}{\left(\sqrt{x+1}+2\right)}=\lim_{x\to 3}\frac{x+1-4}{(x-3)\left(\sqrt{x+1}+2\right)}$

$=\lim_{x\to 3}\frac{x-3}{(x-3)\left(\sqrt{x+1}+2\right)}=\lim_{x\to 3}\frac{1}{\sqrt{x+1}+2}=\frac{1}{\sqrt{3+1}+2}=\frac{1}{4}$

c. $\lim_{x\to 3}\sqrt{x+1}=\sqrt{3+1}=2$

d. This is an infinite limit, and $5-x>0$ as $x\to 5^-$, so $\lim_{x\to 5^-}\frac{8}{5-x}=\infty$.

e. Since $x\to 3^+$, $x>3$, so $3-x<0$ and then $[\![3-x]\!]\to -1$. Therefore, $\lim_{x\to 3^+}\frac{[\![3-x]\!]}{4-x}=-\frac{1}{4-3}=-1$.

3.

$\lim_{x\to 4}\frac{\frac{x}{x+1}-\frac{4}{5}}{x-4}$

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$	0.04082	0.04008	0.04001	0.03999	0.03992	0.03922

So our estimate is $\lim_{x\to 4}\frac{\frac{x}{x+1}-\frac{4}{5}}{x-4}=0.04$ or $\frac{1}{25}$.

4.

$$\lim_{x\to 2}(4x-3)=5$$

To find δ, we first start with $|f(x)-L|<0.01$.

$$\begin{aligned} & |f(x)-L|<0.01 \\ \Leftrightarrow\ & |(4x-3)-5|<0.01 \\ \Leftrightarrow\ & |4x-8|<0.01 \\ \Leftrightarrow\ & 4|x-2|<0.01 \\ \Leftrightarrow\ & |x-2|<0.0025 \end{aligned}$$

So $\delta=0.0025$. To show this δ makes the statement true, we just work backward.

If $|x-2|<0.0025$, then $|f(x)-L|=|(4x-3)-5|=|4x-8|=4|x-2|<4\cdot 0.0025=0.01$.

5. $\displaystyle\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=3$

6. Let $f(x)=x^3+x-1$. Since *f* is a polynomial, it is continuous and satisfies the Intermediate Value Theorem. Also, $f(0)=-1<0$ and $f(1)=1>0$, so by the IVT, there must be a value *c* in the interval $(0,1)$ where $f(c)=0$, i.e. $x^3+x-1=0$.

7. $f(x)=[[x]]$ is not continuous over the interval $(0,1)$, so it does not satisfy the IVT.

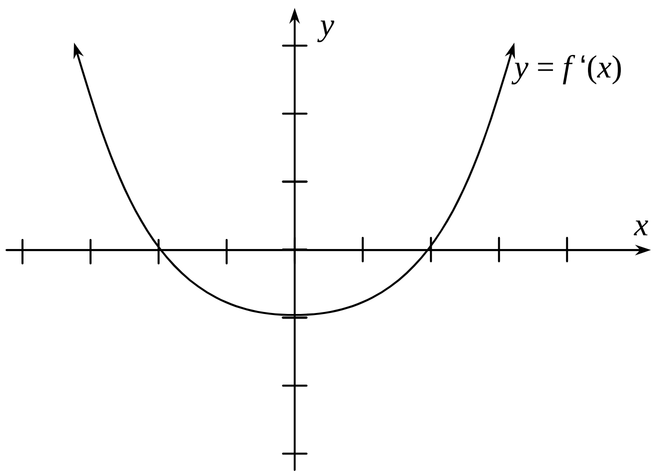
8. *f* has vertical asymptotes when the denominator is zero, but the numerator is not.

   $x^2+7x+10=0 \quad\Rightarrow\quad (x+2)(x+5)=0 \quad\Rightarrow\quad x=-2,-5$, but $f(-2)=\frac{0}{0}$, so only the line $x=-5$ is a vertical asymptote.

9.

$$\begin{aligned} f'(x) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{1}{x+\Delta x}-\frac{1}{x}}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{\left(\frac{1}{x+\Delta x}-\frac{1}{x}\right)}{\Delta x}\cdot\frac{x(x+\Delta x)}{x(x+\Delta x)}=\lim_{\Delta x\to 0}\frac{x-(x+\Delta x)}{\Delta x\cdot x(x+\Delta x)} \\ &= \lim_{\Delta x\to 0}\frac{-\Delta x}{\Delta x\cdot x(x+\Delta x)}=\lim_{\Delta x\to 0}\frac{-1}{x(x+\Delta x)}=\frac{-1}{x(x+0)}=-\frac{1}{x^2} \end{aligned}$$

10.



11. The arrow reaches its maximum height when its velocity is zero, so:

$$v(t)=\frac{dH}{dt}=58-1.66t=0 \quad \Rightarrow \quad t=\frac{58}{1.66}\approx 34.93\,\text{sec}.$$

12. There are many different correct answers to this problem, but there are only two primary key characteristics – either the function has a cusp (corner) or a vertical tangent line.

13.

a. $\frac{dy}{dx}=\pi \cdot x^{\pi-1}$

b. (Use the chain rule.) $g'(x)=5(x^2-3)^4(2x)$

c. (Use the chain rule *twice*.) $y=\sin(\cos^2 x)=\sin\left[(\cos x)^2\right]$, so

$$\frac{dy}{dx}=\cos\left[(\cos x)^2\right](2\cos x)(-\sin x)=-2\cos x\sin x\cos(\cos^2 x)$$

d. $\frac{dy}{dx}=15x^4+6x^2-12x$

e. (Use the product rule.) $f'(x)=\sec x\tan x\csc x+\sec x(-\csc x\cot x)=\sec^2 x-\csc^2 x$

f. (Use the quotient rule.) $\frac{dy}{dx}=\frac{x^2\sec x\tan x-\sec x(2x)}{(x^2)^2}=\frac{\sec x(x\tan x-2)}{x^3}$

14. We first need the slope of the tangent line, so we find the derivative at the point $(\pi, 0)$. $y' = 2\cos 2x$ (chain rule), so $m = y'(\pi) = 2\cos(2\pi) = 2\cdot 1 = 2$. The equation for the line is then $y - y_1 = m(x - x_1) \quad \Rightarrow \quad y - 0 = 2(x - \pi) \quad \Rightarrow \quad y = 2x - 2\pi$.

15.

$$\frac{d}{dx}\left(x^2 y + x^3 y^3\right) = \frac{d}{dx}(4)$$

$$\frac{d}{dx}\left(x^2 y\right) + \frac{d}{dx}\left(x^3 y^3\right) = \frac{d}{dx}(4)$$

$$2xy + x^2 \tfrac{dy}{dx} + 3x^2 y^3 + x^3 \cdot 3y^2 \tfrac{dy}{dx} = 0$$

$$x^2 \tfrac{dy}{dx} + 3x^3 y^2 \tfrac{dy}{dx} = -2xy - 3x^2 y^3$$

$$\tfrac{dy}{dx}\left(x^2 + 3x^3 y^2\right) = -2xy - 3x^2 y^3$$

$$\frac{dy}{dx} = \frac{-2xy - 3x^2 y^3}{x^2 + 3x^3 y^2}$$

16. We need to first find $\frac{dy}{dx}$, and then take the derivative of our solution.

$$x^2 - y^2 = 16$$

$$\frac{d}{dx}\left(x^2 - y^2\right) = \frac{d}{dx}(16)$$

$$2x - 2y\tfrac{dy}{dx} = 0$$

$$-2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{x}{y}$$

Then $\dfrac{d^2 y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}\left(\dfrac{x}{y}\right) = \dfrac{y \cdot 1 - x \cdot \frac{dy}{dx}}{y^2} = \dfrac{y - x \cdot \frac{x}{y}}{y^2} = \dfrac{y - \frac{x^2}{y}}{y^2} \cdot \dfrac{y}{y} = \dfrac{y^2 - x^2}{y^3}$

17. This is actually very similar to problem #27 from section 2.6. You should first draw a picture to get an idea of what the situation is. The equation you'll be working with is $a^2 + b^2 = c^2$ (or some similar version of the Pythagorean Theorem). We are given that the base is 12 feet from the foot of the wall, the ladder is 13 feet long, and that the base is moving away from the wall at as rate of 5 ft/sec. If *a* is the distance of the base of the ladder from the wall, we have the following given information:
$a = 12$ ft and $\frac{da}{dt} = 5$ ft/sec (Note that $\frac{da}{dt}$ is positive since the ladder is moving away from the wall, so the distance is *increasing*.)
Using the Pythagorean Theorem again, we can also find that $b = 5$ ft. The information we are looking for is how fast the ladder is sliding down the wall, which would be $\frac{db}{dt}$ according to my choices. Our next step is to differentiate the original equation and to solve for $\frac{db}{dt}$.
$a^2 + b^2 = c^2 \quad \Rightarrow \quad a^2 + b^2 = 13^2$ (We can insert 13 for *c*, since it is *constant*.)
differentiating:
$$\frac{d}{dt}\left(a^2 + b^2\right) = \frac{d}{dt}\left(13^2\right) \quad \Rightarrow \quad 2a\frac{da}{dt} + 2b\frac{db}{dt} = 0 \quad \Rightarrow \quad \frac{db}{dt} = -\frac{a}{b}\frac{da}{dt} = -\frac{12}{5}5 = -12 \text{ ft/sec}.$$
The answer, then, is that the top of the ladder is sliding down the wall at a rate of 12 ft/sec.

18. Critical numbers are where the derivative is zero or undefined.
$$f'(x) = \frac{\left(x^2+1\right)(4) - (4x)(2x)}{\left(x^2+1\right)^2} = \frac{4-4x^2}{\left(x^2+1\right)^2}$$
The derivative is never undefined, since $\left(x^2+1\right)^2$ is never zero, so the only critical numbers are when $f'(x) = 0 \quad \Rightarrow \quad 4 - 4x^2 = 0 \quad \Rightarrow \quad 4 = 4x^2 \quad \Rightarrow \quad 1 = x^2 \quad \Rightarrow \quad x = \pm 1$.

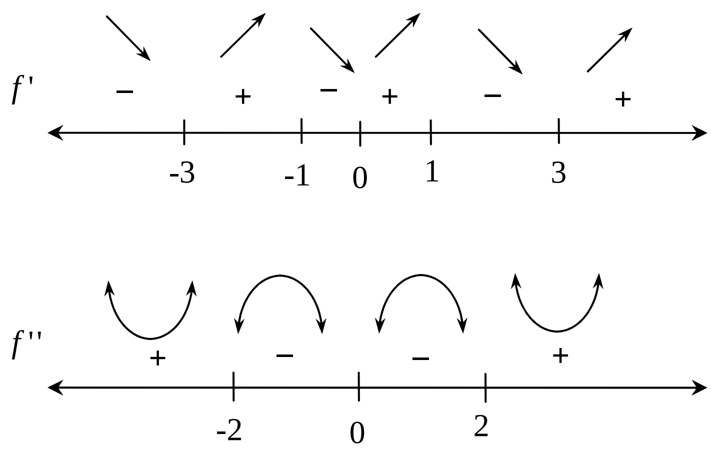
19. Using the first derivative test, we see that the critical numbers are at $x = \pm 1$. Using $x = -2, 0, \& \ 2$ as test points, we see that *f* has a relative maximum at $(-1, 4)$ and a relative minimum at $(1, -4)$.

20. Since *f* is a root function, it is continuous and differentiable on the interval $[1, 9]$. To find the value(s) for *c*, we solve the given equation.
$$f'(c) = \frac{f(9) - f(1)}{9-1} \quad \Leftrightarrow \quad \frac{1}{2\sqrt{c}} = \frac{3-1}{9-1} \quad \Leftrightarrow \quad \frac{1}{2\sqrt{c}} = \frac{1}{4} \quad \Leftrightarrow \quad 4 = 2\sqrt{c} \quad \Leftrightarrow \quad c = 4$$

21. Again using the first derivative test, we can find that
$$f'(x) = \frac{2}{3}\left(x^2 - 4\right)^{-\frac{1}{3}}(2x) = \frac{4x}{3\left(x^2-4\right)^{\frac{1}{3}}} \text{ (chain rule).}$$
The critical numbers are thus $x = 0, \pm 2$. Using test values of $x = -3, -1, 1, \& \ 3$, we can see that *f* has a relative maximum at $x = 0$ and relative minimums at $x = \pm 2$.

22. $f'(x) = 3x^2 - 12x = 3x(x-4)$, so the critical numbers are $x = 0, 4$. Using test values and the first derivative, we can see that *f* is increasing on $(-\infty, 0) \cup (4, \infty)$ and decreasing on $(0, 4)$.

23.



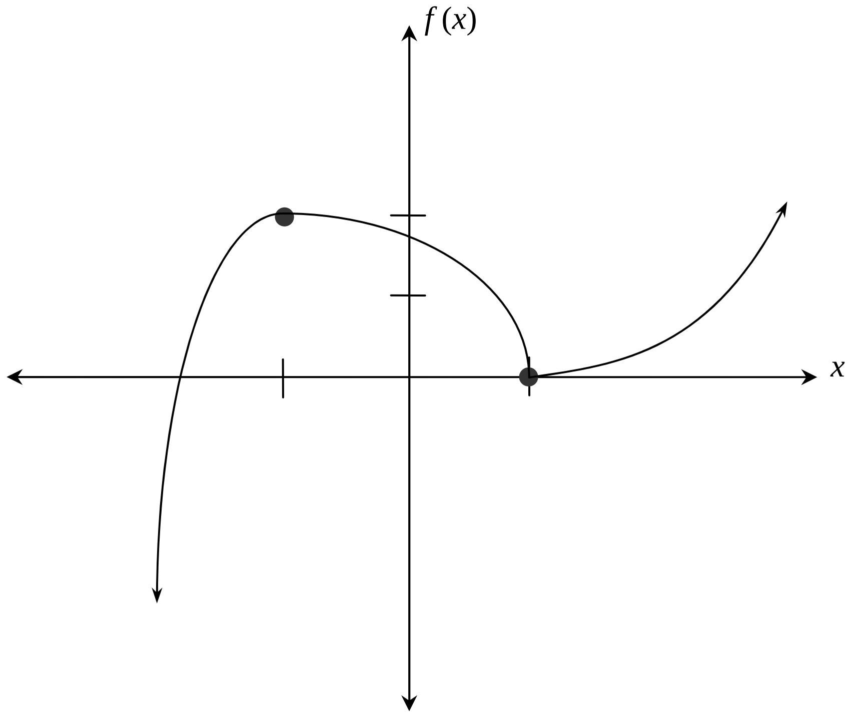
24. Since the second derivative is positive, the graph must be concave up, so the function must have a <u>minimum</u> at $x = 2$.

25. Horizontal asymptotes are where the limit as $x \to \pm\infty$ is a constant.

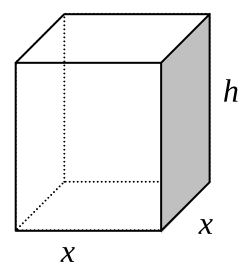
$\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \dfrac{3 - x + 2x^2}{5 + 2x - 4x^2} = -\dfrac{1}{2}$, so the horizontal asymptote is $y = -\dfrac{1}{2}$.

26. $$\lim_{x\to-\infty} \frac{\sqrt{x^2+4x}}{4x+1} = \lim_{x\to-\infty} \frac{\frac{1}{x}}{\frac{1}{x}} \frac{\sqrt{x^2+4x}}{4x+1} = \lim_{x\to-\infty} \frac{-\frac{1}{\sqrt{x^2}}}{\frac{1}{x}} \frac{\sqrt{x^2+4x}}{4x+1} \lim_{x\to-\infty} \frac{-\sqrt{1+\frac{4}{x}}}{4+\frac{1}{x}} = \frac{-\sqrt{1+0}}{4+0} = -\frac{1}{4}$$

27. One possibility:



28. The drawing would look something like the one shown. Because we're looking for the largest volume, we want an equation for the volume in terms of a single variable. In general, the volume of a box is: $V = l \cdot w \cdot h$. In our case, the length and width are the same, so we can have the equation $V = x^2 h$.



To get h in terms of x, we use the given piece of information. Surface area for this box is $1350 = 2x^2 + 4xh$. Solving for h, we get $h = \dfrac{1350 - 2x^2}{4x}$. Substituting that into the equation for volume, we have:

$$V = x^2 \cdot \frac{1350 - 2x^2}{4x} = \frac{1350}{4}x - \frac{1}{2}x^3.$$

To maximize, we take the derivative and find the critical numbers.

$$V'(x) = \frac{1350}{4} - \frac{3}{2}x^2 = 0 \quad \Rightarrow \quad \frac{3}{2}x^2 = \frac{1350}{4} \quad \Rightarrow \quad x^2 = 225 \quad \Rightarrow \quad x = 15$$

(We ignore the negative value here since x represents a length.)

Therefore, the length and width of the box are both 15 cm, and after some calculation, the height is also 15 cm.

29. Because at $x = 1$, the curve $y = x^3 - 3x^2 + 3x + 2$ has a horizontal tangent line.

30. Newton's Method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. To use Newton's Method, we need a function and a starting quess. In this case, a reasonable starting point would be $x_1 = 1$. Since we're trying to find $\sqrt{2}$, we're really trying to solve $x = \sqrt{2} \quad \Rightarrow \quad x^2 = 2$, but Newton's Method is for finding *zeros*, so we should solve $x^2 - 2 = 0$, and let $f(x) = x^2 - 2$. Then $f'(x) = 2x$, and $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n{}^2 - 2}{2x_n}$. Successive iterations are:

$x_1 = 1$
$x_2 = 1.5$
$x_3 = 1.416666667$
$x_4 = 1.414215686$
$x_5 = 1.414213562$
$x_6 = 1.414213562$

So $\sqrt{2} \approx 1.414213562$, accurate to 8 decimal places.

31. To approximate $\sqrt[3]{26}$, we let $f(x) = \sqrt[3]{x}$. We know $\sqrt[3]{27}$, so we use differentials with $x = 27$ and $\Delta x = dx = -1$.

$$dy = \frac{1}{3} x^{-\frac{2}{3}} dx = \frac{dx}{3\sqrt[3]{x^2}} = \frac{-1}{3\left(\sqrt[3]{27}\right)^2} = -\frac{1}{3 \cdot 3^2} = -\frac{1}{27}$$

Therefore, $\sqrt[3]{26} \approx \sqrt[3]{27} + dy = 3 - \frac{1}{27} = \frac{81}{27} - \frac{1}{27} = \frac{80}{27} \approx 2.963$.

32. Using the chain rule, $dy = \frac{1}{2}\left(9 - x^2\right)^{-\frac{1}{2}}(-2x)dx = \frac{-xdx}{\sqrt{9 - x^2}}$

33. The volume of a cylinder in general is $V = \pi \cdot r^2 h$. We want to estimate the change in V when r changes $\frac{1}{4}$ in. Using differentials, we must first find dV and then use the given information.

$$dV = 2\pi \cdot rhdr = 2\pi(12)(34)\left(\tfrac{1}{4}\right) = 204\pi \approx 641 \text{ in}^3.$$

34. Since $y = \int\left(-\frac{1}{x^2}\right)dx = \frac{1}{x} + C$, and the graph passes through the point $(1, 3)$, we know $3 = \frac{1}{1} + C$, so $C = 2$. Therefore, $y = \frac{1}{x} + C$.

35. Find each indefinite integral.

a. $\int \frac{2}{x^3}dx = \int 2x^{-3}dx = 2\cdot\frac{x^{-2}}{-2} + C = -x^{-2} + C = -\frac{1}{x^2} + C$

b. $\int \frac{x+1}{\sqrt{x}}dx = \int\left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right)dx = \int\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)dx = \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

c. $\int (t^2 - \sin t)dt = \frac{1}{3}t^3 + \cos t + C$

36. $\sum_{i=1}^{10}\frac{5}{1+i}$

37.

$$\sum_{i=1}^{n}\frac{(i+1)^2}{n^3} = \sum_{i=1}^{n}\frac{i^2+2i+1}{n^3} = \frac{1}{n^3}\sum_{i=1}^{n}(i^2+2i+1) = \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6} + 2\cdot\frac{n(n+1)}{2} + n\right)$$
$$= \frac{n(n+1)(2n+1)}{6n^3} + \frac{n(n+1)}{n^3} + \frac{1}{n^2}$$

(This can be simplified further, but this point is fine.)

38. $A \approx \left(\tfrac{1}{4}\right)\left(4-\left(\tfrac{5}{4}\right)^2\right) + \left(\tfrac{1}{4}\right)\left(4-\left(\tfrac{3}{2}\right)^2\right) + \left(\tfrac{1}{4}\right)\left(4-\left(\tfrac{7}{4}\right)^2\right) + \left(\tfrac{1}{4}\right)\left(4-2^2\right) = \frac{41}{32} = 1.28125$ units^2.

39. $\Delta x = \frac{3-0}{n} = \frac{3}{n}$. Using right-endpoints, $x_i = \frac{3i}{n}$ and $f(x_i) = \left(\frac{3i}{n}\right)^2 + 1 = \frac{9i^2}{n^2} + 1$. The area is:

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{9i^2}{n^2}+1\right)\cdot\frac{3}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{27i^2}{n^3}+\frac{3}{n}\right) = \lim_{n\to\infty}\left(\frac{27}{n^3}\sum_{i=1}^{n} i^2 + \frac{3}{n}\sum_{i=1}^{n}1\right)$$
$$= \lim_{n\to\infty}\left(\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}\cdot n\right) = \frac{54}{6} + 3 = 12$$

40. (Same problem idea as #39.)

$\Delta x = \frac{2-0}{n} = \frac{2}{n}$. Using right endpoints, $x_i = \frac{2i}{n}$ and $f(x_i) = \left(\frac{2i}{n}\right)^3 + \frac{2i}{n} = \frac{8i^3}{n^3} + \frac{2i}{n}$.

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{8i^3}{n^3}+\frac{2i}{n}\right)\cdot\frac{2}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{16i^3}{n^4}+\frac{4i}{n^2}\right) = \lim_{n\to\infty}\left(\frac{16}{n^4}\sum_{i=1}^{n} i^3 + \frac{4}{n^2}\sum_{i=1}^{n} i\right)$$

$$= \lim_{n\to\infty}\left(\frac{16}{n^4}\cdot\frac{n^2(n+1)^2}{4} + \frac{4}{n^2}\cdot\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\left(\frac{16n^2(n+1)^2}{4n^4} + \frac{4n(n+1)}{2n^2}\right) = 4+2=6$$

41.

a. $\int \left(4x^3+6x^2-1\right)dx = x^4+2x^3-x+C$

b. Use substitution, with $u=\sin y$, then $du=\cos y\, dy$, and the integral then becomes:

$$\int \cos y\sqrt{\sin y}\,dy = \int \sqrt{u}\,du = \int u^{\frac{1}{2}}du = 2u^{\frac{3}{2}}+C = 2(\sin y)^{\frac{3}{2}}+C$$

c. The function $f(x) = \frac{x^2}{\tan x}$ has odd symmetry, since

$f(-x) = \frac{(-x)^2}{\tan(-x)} = \frac{x^2}{-\tan x} = -f(x)$. Therefore, $\int_{-\pi}^{\pi}\frac{x^2}{\tan x}dx = 0$.

d. Use substitution again, with $u=1+x^4$, so $du=4x^3dx \quad\Rightarrow\quad x^3dx=\frac{1}{4}du$. If we change the bounds, we see that

$$\int_0^1 \frac{x^3}{\left(1+x^4\right)^3}dx = \frac{1}{4}\int_1^2\frac{1}{u^3}du = \frac{1}{4}\int_1^2 u^{-3}du = -\frac{1}{8}\frac{1}{u^2}\Big|_1^2 = -\frac{1}{8}\left(\frac{1}{4}-1\right) = -\frac{1}{8}\left(-\frac{3}{4}\right) = \frac{3}{32}$$

42. (4.6) Use the Trapezoid rule to estimate the number of square meters of land in a lot where x and y are measured in meters, as shown. The land is bounded by a stream and two straight roads that meet at right angles.

x	y
0	58
10	62
20	59
30	51
40	50
50	52
60	41
70	38
80	28
90	14
100	0