Mth 114 – Trigonometry – **Practice Exam 3 Solutions**

Half-Angle Identities:	Product-to-Sum Identities:	Sum-to-Product Identities:
$\cos\frac{A}{2}=\pm\sqrt{\frac{1+\cos A}{2}}$	$\cos A\cos B=\frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right]$	$\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
$\sin\frac{A}{2}=\pm\sqrt{\frac{1-\cos A}{2}}$	$\sin A\sin B=\frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$	$\sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$
$\tan\frac{A}{2}=\pm\sqrt{\frac{1-\cos A}{1+\cos A}}$	$\sin A\cos B=\frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right]$	$\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
$\tan\frac{A}{2}=\frac{\sin A}{1+\cos A}=\frac{1-\cos A}{\sin A}$	$\cos A\sin B=\frac{1}{2}\left[\sin(A+B)-\sin(A-B)\right]$	$\cos A-\cos B=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

1. Find $\sin s$ given $\cos s=\frac{3}{5}$ and s is in quadrant IV.

Using the Pythagorean identity,

$$\sin^2 s+\cos^2 s=1$$

$$\sin^2 s+\left(\frac{3}{5}\right)^2=1 \quad\Rightarrow\quad \sin^2 s+\frac{9}{25}=1$$

$$\sin^2 s=1-\frac{9}{25}=\frac{16}{25} \quad\Rightarrow\quad \sin s=-\frac{4}{5}$$

(Remember, sine is negative in quadrant IV.)

2. Find $\cos(s+t)$ given that $\sin s=\frac{2}{3}$ and $\sin t=-\frac{1}{3}$, s in quadrant II and t in quadrant IV.

We know $\cos(s+t)=\cos s\cos t-\sin s\sin t$, so we need to find cos s and cos t.

Using the Pythagorean identity,

$$\sin^2 s+\cos^2 s=1 \qquad\qquad \sin^2 t+\cos^2 t=1$$

$$\left(\frac{2}{3}\right)^2+\cos^2 s=1 \quad\Rightarrow\quad \frac{4}{9}+\cos^2 s=1 \qquad\qquad \left(-\frac{1}{3}\right)^2+\cos^2 t=1 \quad\Rightarrow\quad \frac{1}{9}+\cos^2 t=1$$

$$\cos^2 s=1-\frac{4}{9}=\frac{5}{9} \quad\Rightarrow\quad \cos s=-\frac{\sqrt{5}}{3} \qquad\qquad \cos^2 t=1-\frac{1}{9}=\frac{8}{9} \quad\Rightarrow\quad \cos t=\frac{2\sqrt{2}}{3}$$

So $\cos(s+t)=\cos s\cos t-\sin s\sin t=\left(-\frac{\sqrt{5}}{3}\right)\left(\frac{2\sqrt{2}}{3}\right)-\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)=\frac{-2\sqrt{10}+2}{9}$

3. Verify the following identities.

a. $\dfrac{\sin^2\theta}{\cos\theta} = \sec\theta - \cos\theta$

$$= \frac{1}{\cos\theta} - \cos\theta$$
$$= \frac{1}{\cos\theta} - \frac{\cos^2\theta}{\cos\theta}$$
$$= \frac{1-\cos^2\theta}{\cos\theta}$$
$$= \frac{\sin^2\theta}{\cos\theta}$$

b. $\dfrac{1}{1-\sin\alpha} + \dfrac{1}{1+\sin\alpha} = 2\sec^2\alpha$

$$= \frac{1}{1-\sin\alpha}\cdot\frac{1+\sin\alpha}{1+\sin\alpha} + \frac{1}{1+\sin\alpha}\cdot\frac{1-\sin\alpha}{1-\sin\alpha}$$
$$= \frac{1+\sin\alpha}{1-\sin^2\alpha} + \frac{1-\sin\alpha}{1-\sin^2\alpha}$$
$$= \frac{1+\sin\alpha+1-\sin\alpha}{1-\sin^2\alpha}$$
$$= \frac{2}{\cos^2\alpha} = 2\sec^2\alpha$$

c. $\dfrac{\sin(x-y)}{\sin y} + \dfrac{\cos(x-y)}{\cos y} = \dfrac{\sin x}{\sin y\cos y}$

$$= \frac{\sin x\cos y - \cos x\sin y}{\sin y} + \frac{\cos x\cos y + \sin x\sin y}{\cos y}$$
$$= \frac{\sin x\cos y}{\sin y} - \cos x + \cos x + \frac{\sin x\sin y}{\cos y}$$
$$= \frac{\sin x\cos y}{\sin y} + \frac{\sin x\sin y}{\cos y}$$
$$= \frac{\sin x\cos y}{\sin y}\cdot\frac{\cos y}{\cos y} + \frac{\sin x\sin y}{\cos y}\cdot\frac{\sin y}{\sin y}$$
$$= \frac{\sin x\cos^2 y + \sin x\sin^2 y}{\sin y\cos y}$$
$$= \frac{\sin x\left(\cos^2 y + \sin^2 y\right)}{\sin y\cos y}$$
$$= \frac{\sin x}{\sin y\cos y}$$

d. $\dfrac{2\cos 2\theta}{\sin 2\theta} = \cot\theta - \tan\theta$

$$= \frac{2\left(\cos^2\theta - \sin^2\theta\right)}{2\sin\theta\cos\theta}$$
$$= \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}$$
$$= \frac{\cos^2\theta}{\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta}$$
$$= \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$$
$$= \cot\theta - \tan\theta$$

4. Use identities to find each exact value.

a. $\sin 75°$

There is more than one right way to do this one. I chose:

$$\sin 75° = \sin\frac{150°}{2} = \pm\sqrt{\frac{1-\cos 150°}{2}} = \sqrt{\frac{1 - -\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}\cdot\frac{2}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}$$

Another option would have been $\sin 75° = \sin\left(45° + 30°\right)$

b. $\tan\dfrac{\pi}{12}$

There's also more than one right way here.

$$\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1+\tan\frac{\pi}{3}\tan\frac{\pi}{4}} = \frac{\sqrt{3}-1}{1+\sqrt{3}\cdot 1} = \frac{\sqrt{3}-1}{1+\sqrt{3}}\cdot\frac{1-\sqrt{3}}{1-\sqrt{3}} = \frac{\sqrt{3}-3-1+\sqrt{3}}{1-3}$$
$$= \frac{2\sqrt{3}-4}{-2} = -\sqrt{3}+2 = 2-\sqrt{3}$$

c. $\sin 22.5°$

There's only one option here.

$$\sin 22.5° = \sin\frac{45°}{2} = \pm\sqrt{\frac{1-\cos 45°}{2}} = \pm\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}\cdot\frac{2}{2}} = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

5. Find $\cos 2\theta$ given $\cos\theta = -\dfrac{12}{13}$.

$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\cos 2\theta = 2\left(-\frac{12}{13}\right)^2 - 1 = 2\cdot\frac{144}{169} - 1 = \frac{288}{169} - \frac{169}{169} = \frac{119}{169}$$

6. Write the expression $\sin 5x\cos 3x$ as a sum or difference of trigonometric functions.

$$\sin 5x\cos 3x = \tfrac{1}{2}\left[\sin(5x+3x) + \sin(5x-3x)\right] = \tfrac{1}{2}\sin 8x + \tfrac{1}{2}\sin 2x$$

7. Find $\cos\dfrac{\theta}{2}$, given $\sin\theta = -\dfrac{1}{5}$, with $180° < \theta < 270°$.

Since $\cos\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$, we first need to find $\cos\theta$.

Using the Pythagorean identity,

$$\sin^2\theta + \cos^2\theta = 1$$
$$\left(-\frac{1}{5}\right)^2 + \cos^2\theta = 1 \quad\Rightarrow\quad \frac{1}{25} + \cos^2\theta = 1$$
$$\cos^2\theta = 1 - \frac{1}{25} = \frac{24}{25} \quad\Rightarrow\quad \cos\theta = -\frac{2\sqrt{6}}{5}$$

Then,

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} = \pm\sqrt{\frac{1-\frac{2\sqrt{6}}{5}}{2}} = \pm\sqrt{\frac{1-\frac{2\sqrt{6}}{5}}{2}\cdot\frac{5}{5}} = \pm\sqrt{\frac{5-2\sqrt{6}}{10}}^{\,*}$$
$$= \pm\frac{\sqrt{5-2\sqrt{6}}}{\sqrt{10}} = \pm\frac{\sqrt{10}\sqrt{5-2\sqrt{6}}}{10} = \pm\frac{\sqrt{50-20\sqrt{6}}}{10}$$

***** This expression technically isn't simplified, but the rest of the steps get pretty messy.

Since, $180° < \theta < 270°$, $90° < \dfrac{\theta}{2} < 135°$, so $\cos\dfrac{\theta}{2}$ must be negative.

Therefore, $\cos\dfrac{\theta}{2} = -\dfrac{\sqrt{50-20\sqrt{6}}}{10}$.