College Trigonometry
Version $\lfloor \pi \rfloor$
Corrected Edition

by

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Acknowledgements

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Thank you for your interest in our book, but more importantly, thank you for taking the time to read the Preface. I always read the Prefaces of the textbooks which I use in my classes because I believe it is in the Preface where I begin to understand the authors - who they are, what their motivation for writing the book was, and what they hope the reader will get out of reading the text. Pedagogical issues such as content organization and how professors and students should best use a book can usually be gleaned out of its Table of Contents, but the reasons behind the choices authors make should be shared in the Preface. Also, I feel that the Preface of a textbook should demonstrate the authors’ love of their discipline and passion for teaching, so that I come away believing that they really want to help students and not just make money. Thus, I thank my fellow Preface-readers again for giving me the opportunity to share with you the need and vision which guided the creation of this book and passion which both Carl and I hold for Mathematics and the teaching of it.

Carl and I are natives of Northeast Ohio. We met in graduate school at Kent State University in 1997. I finished my Ph.D in Pure Mathematics in August 1998 and started teaching at Lorain County Community College in Elyria, Ohio just two days after graduation. Carl earned his Ph.D in Pure Mathematics in August 2000 and started teaching at Lakeland Community College in Kirtland, Ohio that same month. Our schools are fairly similar in size and mission and each serves a similar population of students. The students range in age from about 16 (Ohio has a Post-Secondary Enrollment Option program which allows high school students to take college courses for free while still in high school.) to over 65. Many of the “non-traditional” students are returning to school in order to change careers. A majority of the students at both schools receive some sort of financial aid, be it scholarships from the schools' foundations, state-funded grants or federal financial aid like student loans, and many of them have lives busied by family and job demands. Some will be taking their Associate degrees and entering (or re-entering) the workforce while others will be continuing on to a four-year college or university. Despite their many differences, our students share one common attribute: they do not want to spend $200 on a College Algebra book.

The challenge of reducing the cost of textbooks is one that many states, including Ohio, are taking quite seriously. Indeed, state-level leaders have started to work with faculty from several of the colleges and universities in Ohio and with the major publishers as well. That process will take considerable time so Carl and I came up with a plan of our own. We decided that the best way to help our students right now was to write our own College Algebra book and give it away electronically for free. We were granted sabbaticals from our respective institutions for the Spring
semester of 2009 and actually began writing the textbook on December 16, 2008. Using an open-source text editor called TexNicCenter and an open-source distribution of LaTeX called MikTex 2.7, Carl and I wrote and edited all of the text, exercises and answers and created all of the graphs (using Metapost within LaTeX) for Version 0.9 in about eight months. (We choose to create a text in only black and white to keep printing costs to a minimum for those students who prefer a printed edition. This somewhat Spartan page layout stands in sharp relief to the explosion of colors found in most other College Algebra texts, but neither Carl nor I believe the four-color print adds anything of value.) I used the book in three sections of College Algebra at Lorain County Community College in the Fall of 2009 and Carl’s colleague, Dr. Bill Previts, taught a section of College Algebra at Lakeland with the book that semester as well. Students had the option of downloading the book as a .pdf file from our website www.stitz-zeager.com or buying a low-cost printed version from our colleges’ respective bookstores. (By giving this book away for free electronically, we end the cycle of new editions appearing every 18 months to curtail the used book market.) During Thanksgiving break in November 2009, many additional exercises written by Dr. Previts were added and the typographical errors found by our students and others were corrected. On December 10, 2009, Version $\sqrt{2}$ was released. The book remains free for download at our website and by using Lulu.com as an on-demand printing service, our bookstores are now able to provide a printed edition for just under $19. Neither Carl nor I have, or will ever, receive any royalties from the printed editions. As a contribution back to the open-source community, all of the LaTeX files used to compile the book are available for free under a Creative Commons License on our website as well. That way, anyone who would like to rearrange or edit the content for their classes can do so as long as it remains free.

The only disadvantage to not working for a publisher is that we don’t have a paid editorial staff. What we have instead, beyond ourselves, is friends, colleagues and unknown people in the open-source community who alert us to errors they find as they read the textbook. What we gain in not having to report to a publisher so dramatically outweighs the lack of the paid staff that we have turned down every offer to publish our book. (As of the writing of this Preface, we’ve had three offers.) By maintaining this book by ourselves, Carl and I retain all creative control and keep the book our own. We control the organization, depth and rigor of the content which means we can resist the pressure to diminish the rigor and homogenize the content so as to appeal to a mass market. A casual glance through the Table of Contents of most of the major publishers’ College Algebra books reveals nearly isomorphic content in both order and depth. Our Table of Contents shows a different approach, one that might be labeled “Functions First.” To truly use The Rule of Four, that is, in order to discuss each new concept algebraically, graphically, numerically and verbally, it seems completely obvious to us that one would need to introduce functions first. (Take a moment and compare our ordering to the classic “equations first, then the Cartesian Plane and THEN functions” approach seen in most of the major players.) We then introduce a class of functions and discuss the equations, inequalities (with a heavy emphasis on sign diagrams) and applications which involve functions in that class. The material is presented at a level that definitely prepares a student for Calculus while giving them relevant Mathematics which can be used in other classes as well. Graphing calculators are used sparingly and only as a tool to enhance the Mathematics, not to replace it. The answers to nearly all of the computational homework exercises are given in the
text and we have gone to great lengths to write some very thought provoking discussion questions whose answers are not given. One will notice that our exercise sets are much shorter than the traditional sets of nearly 100 “drill and kill” questions which build skill devoid of understanding. Our experience has been that students can do about 15-20 homework exercises a night so we very carefully chose smaller sets of questions which cover all of the necessary skills and get the students thinking more deeply about the Mathematics involved.

Critics of the Open Educational Resource movement might quip that “open-source is where bad content goes to die,” to which I say this: take a serious look at what we offer our students. Look through a few sections to see if what we’ve written is bad content in your opinion. I see this open-source book not as something which is “free and worth every penny”, but rather, as a high quality alternative to the business as usual of the textbook industry and I hope that you agree. If you have any comments, questions or concerns please feel free to contact me at jeff@stitz-zeager.com or Carl at carl@stitz-zeager.com.

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January 25, 2010
Chapter 10

Foundations of Trigonometry

10.1 Angles and their Measure

This section begins our study of Trigonometry and to get started, we recall some basic definitions from Geometry. A ray is usually described as a ‘half-line’ and can be thought of as a line segment in which one of the two endpoints is pushed off infinitely distant from the other, as pictured below. The point from which the ray originates is called the initial point of the ray.

A ray with initial point $P$.

When two rays share a common initial point they form an angle and the common initial point is called the vertex of the angle. Two examples of what are commonly thought of as angles are

An angle with vertex $P$.

An angle with vertex $Q$.

However, the two figures below also depict angles - albeit these are, in some sense, extreme cases. In the first case, the two rays are directly opposite each other forming what is known as a straight angle; in the second, the rays are identical so the ‘angle’ is indistinguishable from the ray itself.

A straight angle.

The measure of an angle is a number which indicates the amount of rotation that separates the rays of the angle. There is one immediate problem with this, as pictured below.
Which amount of rotation are we attempting to quantify? What we have just discovered is that we have at least two angles described by this diagram. Clearly these two angles have different measures because one appears to represent a larger rotation than the other, so we must label them differently. In this book, we use lower case Greek letters such as $\alpha$ (alpha), $\beta$ (beta), $\gamma$ (gamma) and $\theta$ (theta) to label angles. So, for instance, we have

One commonly used system to measure angles is **degree measure**. Quantities measured in degrees are denoted by the familiar ‘$^\circ$’ symbol. One complete revolution as shown below is $360^\circ$, and parts of a revolution are measured proportionately. Thus half of a revolution (a straight angle) measures $\frac{1}{2} (360^\circ) = 180^\circ$, a quarter of a revolution (a **right angle**) measures $\frac{1}{4} (360^\circ) = 90^\circ$ and so on.

Note that in the above figure, we have used the small square ‘□’ to denote a right angle, as is commonplace in Geometry. Recall that if an angle measures strictly between $0^\circ$ and $90^\circ$ it is called an **acute angle** and if it measures strictly between $90^\circ$ and $180^\circ$ it is called an **obtuse angle**. It is important to note that, theoretically, we can know the measure of any angle as long as we

---

1. The phrase ‘at least’ will be justified in short order.
2. The choice of ‘360’ is most often attributed to the [Babylonians](https://www.britannica.com/technology/babylonian-mathematics).
know the proportion it represents of entire revolution. For instance, the measure of an angle which represents a rotation of \( \frac{2}{3} \) of a revolution would measure \( \frac{2}{3} (360^\circ) = 240^\circ \), the measure of an angle which constitutes only \( \frac{1}{12} \) of a revolution measures \( \frac{1}{12} (360^\circ) = 30^\circ \) and an angle which indicates no rotation at all is measured as \( 0^\circ \).

Using our definition of degree measure, we have that \( 1^\circ \) represents the measure of an angle which constitutes \( \frac{1}{360} \) of a revolution. Even though it may be hard to draw, it is nonetheless not difficult to imagine an angle with measure smaller than \( 1^\circ \). There are two ways to subdivide degrees. The first, and most familiar, is decimal degrees. For example, an angle with a measure of \( 30.5^\circ \) would represent a rotation halfway between \( 30^\circ \) and \( 31^\circ \), or equivalently, \( \frac{30.5}{360} = \frac{61}{720} \) of a full rotation. This can be taken to the limit using Calculus so that measures like \( \sqrt{2}^\circ \) make sense. The second way to divide degrees is the Degree - Minute - Second (DMS) system. In this system, one degree is divided equally into sixty minutes, and in turn, each minute is divided equally into sixty seconds. In symbols, we write \( 1^\circ = 60' \) and \( 1' = 60'' \), from which it follows that \( 1^\circ = 3600'' \). To convert a measure of \( 42.125^\circ \) to the DMS system, we start by noting that \( 42.125^\circ = 42^\circ + 0.125^\circ \). Converting the partial amount of degrees to minutes, we find \( 0.125^\circ \left( \frac{60'}{1^\circ} \right) = 7.5' = 7' + 0.5' \). Converting the partial amount of minutes to seconds gives \( 0.5' \left( \frac{60''}{1'} \right) = 30'' \). Putting it all together yields

\[
42.125^\circ = 42^\circ + 0.125^\circ \\
= 42^\circ + 7.5' \\
= 42^\circ + 7' + 0.5' \\
= 42^\circ + 7' + 30''
\]

On the other hand, to convert \( 117^\circ 15' 45'' \) to decimal degrees, we first compute \( 15' \left( \frac{1^\circ}{60'} \right) = \frac{1}{4}^\circ \) and \( 45'' \left( \frac{1^\circ}{3600''} \right) = \frac{1}{80}^\circ \). Then we find

---

\(^3\)This is how a protractor is graded. \\
\(^4\)Awesome math pun aside, this is the same idea behind defining irrational exponents in Section 6.1. \\
\(^5\)Does this kind of system seem familiar?
Recall that two acute angles are called **complementary angles** if their measures add to 90°. Two angles, either a pair of right angles or one acute angle and one obtuse angle, are called **supplementary angles** if their measures add to 180°. In the diagram below, the angles \( \alpha \) and \( \beta \) are supplementary angles while the pair \( \gamma \) and \( \theta \) are complementary angles.

In practice, the distinction between the angle itself and its measure is blurred so that the sentence ‘\( \alpha \) is an angle measuring 42°’ is often abbreviated as ‘\( \alpha = 42° \).’ It is now time for an example.

**Example 10.1.1.** Let \( \alpha = 111.371° \) and \( \beta = 37°28'17'' \).

1. Convert \( \alpha \) to the DMS system. Round your answer to the nearest second.
2. Convert \( \beta \) to decimal degrees. Round your answer to the nearest thousandth of a degree.
3. Sketch \( \alpha \) and \( \beta \).
4. Find a supplementary angle for \( \alpha \).
5. Find a complementary angle for \( \beta \).

**Solution.**

1. To convert \( \alpha \) to the DMS system, we start with \( 111.371° = 111° + 0.371° \). Next we convert \( 0.371° \left( \frac{60'}{1°} \right) = 22.26' \). Writing 22.26' = 22' + 0.26', we convert \( 0.26' \left( \frac{60''}{1'} \right) = 15.6'' \). Hence,

\[
111.371° = 111° + 0.371° \\
= 111° + 22.26' \\
= 111° + 22' + 0.26' \\
= 111° + 22' + 15.6'' \\
= 111°22'15.6''
\]

Rounding to seconds, we obtain \( \alpha \approx 111°22'16'' \).
2. To convert $\beta$ to decimal degrees, we convert $28' \left( \frac{1^\circ}{15'} \right)$ and $17'' \left( \frac{1^\circ}{3600'} \right)$. Putting it all together, we have

\[
37^\circ 28'17'' = 37^\circ + 28' + 17''
\]
\[
= 37^\circ + \frac{7}{15}^\circ + \frac{17}{3600}^\circ
\]
\[
= \frac{134897}{3600}^\circ
\]
\[
\approx 37.471^\circ
\]

3. To sketch $\alpha$, we first note that $90^\circ < \alpha < 180^\circ$. If we divide this range in half, we get $90^\circ < \alpha < 135^\circ$, and once more, we have $90^\circ < \alpha < 112.5^\circ$. This gives us a pretty good estimate for $\alpha$, as shown below. Proceeding similarly for $\beta$, we find $0^\circ < \beta < 90^\circ$, then $0^\circ < \beta < 45^\circ$, $22.5^\circ < \beta < 45^\circ$, and lastly, $33.75^\circ < \beta < 45^\circ$.

4. To find a supplementary angle for $\alpha$, we seek an angle $\theta$ so that $\alpha + \theta = 180^\circ$. We get $\theta = 180^\circ - \alpha = 180^\circ - 111.371^\circ = 68.629^\circ$.

5. To find a complementary angle for $\beta$, we seek an angle $\gamma$ so that $\beta + \gamma = 90^\circ$. We get $\gamma = 90^\circ - \beta = 90^\circ - 37^\circ 28'17''$. While we could reach for the calculator to obtain an approximate answer, we choose instead to do a bit of sexagesimal arithmetic. We first rewrite $90^\circ = 90^\circ 0'0'' = 89^\circ 60'0'' = 89^\circ 59'60''$. In essence, we are ‘borrowing’ $1^\circ = 60'$ from the degree place, and then borrowing $1' = 60''$ from the minutes place. This yields, $\gamma = 90^\circ - 37^\circ 28'17'' = 89^\circ 59'60'' - 37^\circ 28'17'' = 52^\circ 31'43''$.

Up to this point, we have discussed only angles which measure between $0^\circ$ and $360^\circ$, inclusive. Ultimately, we want to use the arsenal of Algebra which we have stockpiled in Chapters 1 through 9 to not only solve geometric problems involving angles, but also to extend their applicability to other real-world phenomena. A first step in this direction is to extend our notion of ‘angle’ from merely measuring an extent of rotation to quantities which can be associated with real numbers. To that end, we introduce the concept of an oriented angle. As its name suggests, in an oriented

\[\text{If this process seems hauntingly familiar, it should. Compare this method to the Bisection Method introduced in Section 3.3.}\]

\[\text{Like ‘latus rectum,’ this is also a real math term.}\]

\[\text{This is the exact same kind of ‘borrowing’ you used to do in Elementary School when trying to find 300 – 125. Back then, you were working in a base ten system; here, it is base sixty.}\]
angle, the direction of the rotation is important. We imagine the angle being swept out starting from an initial side and ending at a terminal side, as shown below. When the rotation is counter-clockwise\(^9\) from initial side to terminal side, we say that the angle is positive; when the rotation is clockwise, we say that the angle is negative.

![Diagram showing positive and negative angles](image)

A positive angle, \(45^\circ\)  
A negative angle, \(-45^\circ\)

At this point, we also extend our allowable rotations to include angles which encompass more than one revolution. For example, to sketch an angle with measure \(450^\circ\) we start with an initial side, rotate counter-clockwise one complete revolution (to take care of the ‘first’ \(360^\circ\)) then continue with an additional \(90^\circ\) counter-clockwise rotation, as seen below.

![Diagram showing 450° angle](image)

450°

To further connect angles with the Algebra which has come before, we shall often overlay an angle diagram on the coordinate plane. An angle is said to be in standard position if its vertex is the origin and its initial side coincides with the positive \(x\)-axis. Angles in standard position are classified according to where their terminal side lies. For instance, an angle in standard position whose terminal side lies in Quadrant I is called a ‘Quadrant I angle’. If the terminal side of an angle lies on one of the coordinate axes, it is called a quadrantal angle. Two angles in standard position are called coterminal if they share the same terminal side.\(^{10}\) In the figure below, \(\alpha = 120^\circ\) and \(\beta = -240^\circ\) are two coterminal Quadrant II angles drawn in standard position. Note that \(\alpha = \beta + 360^\circ\), or equivalently, \(\beta = \alpha - 360^\circ\). We leave it as an exercise to the reader to verify that coterminal angles always differ by a multiple of \(360^\circ\).\(^{11}\) More precisely, if \(\alpha\) and \(\beta\) are coterminal angles, then \(\beta = \alpha + 360^\circ \cdot k\) where \(k\) is an integer.\(^{12}\)

\(^9\)‘widdershins’

\(^{10}\)Note that by being in standard position they automatically share the same initial side which is the positive \(x\)-axis.

\(^{11}\)It is worth noting that all of the pathologies of Analytic Trigonometry result from this innocuous fact.

\(^{12}\)Recall that this means \(k = 0, \pm 1, \pm 2, \ldots\)
Two coterminal angles, \( \alpha = 120^\circ \) and \( \beta = -240^\circ \), in standard position.

**Example 10.1.2.** Graph each of the (oriented) angles below in standard position and classify them according to where their terminal side lies. Find three coterminal angles, at least one of which is positive and one of which is negative.

1. \( \alpha = 60^\circ \)
2. \( \beta = -225^\circ \)
3. \( \gamma = 540^\circ \)
4. \( \phi = -750^\circ \)

**Solution.**

1. To graph \( \alpha = 60^\circ \), we draw an angle with its initial side on the positive \( x \)-axis and rotate counter-clockwise \( \frac{60^\circ}{360^\circ} = \frac{1}{6} \) of a revolution. We see that \( \alpha \) is a Quadrant I angle. To find angles which are coterminal, we look for angles \( \theta \) of the form \( \theta = \alpha + 360^\circ \cdot k \), for some integer \( k \). When \( k = 1 \), we get \( \theta = 60^\circ + 360^\circ = 420^\circ \). Substituting \( k = -1 \) gives \( \theta = 60^\circ - 360^\circ = -300^\circ \). Finally, if we let \( k = 2 \), we get \( \theta = 60^\circ + 720^\circ = 780^\circ \).

2. Since \( \beta = -225^\circ \) is negative, we start at the positive \( x \)-axis and rotate clockwise \( \frac{225^\circ}{360^\circ} = \frac{5}{8} \) of a revolution. We see that \( \beta \) is a Quadrant II angle. To find coterminal angles, we proceed as before and compute \( \theta = -225^\circ + 360^\circ \cdot k \) for integer values of \( k \). We find \( 135^\circ \), \( -585^\circ \) and \( 495^\circ \) are all coterminal with \(-225^\circ\).  

\( \alpha = 60^\circ \) in standard position. \( \beta = -225^\circ \) in standard position.
3. Since $\gamma = 540^\circ$ is positive, we rotate counter-clockwise from the positive $x$-axis. One full revolution accounts for $360^\circ$, with $180^\circ$, or $\frac{1}{2}$ of a revolution remaining. Since the terminal side of $\gamma$ lies on the negative $x$-axis, $\gamma$ is a quadrantal angle. All angles coterminal with $\gamma$ are of the form $\theta = 540^\circ + 360^\circ \cdot k$, where $k$ is an integer. Working through the arithmetic, we find three such angles: $180^\circ$, $-180^\circ$ and $900^\circ$.

4. The Greek letter $\phi$ is pronounced ‘fee’ or ‘fie’ and since $\phi$ is negative, we begin our rotation clockwise from the positive $x$-axis. Two full revolutions account for $720^\circ$, with just $30^\circ$ or $\frac{1}{12}$ of a revolution to go. We find that $\phi$ is a Quadrant IV angle. To find coterminal angles, we compute $\theta = -750^\circ + 360^\circ \cdot k$ for a few integers $k$ and obtain $-390^\circ$, $-30^\circ$ and $330^\circ$.

Note that since there are infinitely many integers, any given angle has infinitely many coterminal angles, and the reader is encouraged to plot the few sets of coterminal angles found in Example 10.1.2 to see this. We are now just one step away from completely marrying angles with the real numbers and the rest of Algebra. To that end, we recall this definition from Geometry.

**Definition 10.1.** The real number $\pi$ is defined to be the ratio of a circle’s circumference to its diameter. In symbols, given a circle of circumference $C$ and diameter $d$,

$$\pi = \frac{C}{d}$$

While Definition 10.1 is quite possibly the ‘standard’ definition of $\pi$, the authors would be remiss if we didn’t mention that buried in this definition is actually a theorem. As the reader is probably aware, the number $\pi$ is a mathematical constant - that is, it doesn’t matter which circle is selected, the ratio of its circumference to its diameter will have the same value as any other circle. While this is indeed true, it is far from obvious and leads to a counterintuitive scenario which is explored in the Exercises. Since the diameter of a circle is twice its radius, we can quickly rearrange the equation in Definition 10.1 to get a formula more useful for our purposes, namely: $2\pi = \frac{C}{r}$
This tells us that for any circle, the ratio of its circumference to its radius is also always constant; in this case the constant is $2\pi$. Suppose now we take a portion of the circle, so instead of comparing the entire circumference $C$ to the radius, we compare some arc measuring $s$ units in length to the radius, as depicted below. Let $\theta$ be the central angle subtended by this arc, that is, an angle whose vertex is the center of the circle and whose determining rays pass through the endpoints of the arc. Using proportionality arguments, it stands to reason that the ratio $\frac{s}{r}$ should also be a constant among all circles, and it is this ratio which defines the radian measure of an angle.

The radian measure of $\theta$ is $\frac{s}{r}$.

To get a better feel for radian measure, we note that an angle with radian measure 1 means the corresponding arc length $s$ equals the radius of the circle $r$, hence $s = r$. When the radian measure is 2, we have $s = 2r$; when the radian measure is 3, $s = 3r$, and so forth. Thus the radian measure of an angle $\theta$ tells us how many ‘radius lengths’ we need to sweep out along the circle to subtend the angle $\theta$.

Since one revolution sweeps out the entire circumference $2\pi r$, one revolution has radian measure $\frac{2\pi r}{r} = 2\pi$. From this we can find the radian measure of other central angles using proportions,
just like we did with degrees. For instance, half of a revolution has radian measure \( \frac{1}{2}(2\pi) = \pi \), a quarter revolution has radian measure \( \frac{1}{4}(2\pi) = \frac{\pi}{2} \), and so forth. Note that, by definition, the radian measure of an angle is a length divided by another length so that these measurements are actually dimensionless and are considered ‘pure’ numbers. For this reason, we do not use any symbols to denote radian measure, but we use the word ‘radians’ to denote these dimensionless units as needed. For instance, we say one revolution measures ‘\( 2\pi \) radians,’ half of a revolution measures ‘\( \pi \) radians,’ and so forth.

As with degree measure, the distinction between the angle itself and its measure is often blurred in practice, so when we write ‘\( \theta = \frac{\pi}{2} \)’, we mean \( \theta \) is an angle which measures \( \frac{\pi}{2} \) radians.\(^{13}\) We extend radian measure to oriented angles, just as we did with degrees beforehand, so that a positive measure indicates counter-clockwise rotation and a negative measure indicates clockwise rotation.\(^{14}\) Much like before, two positive angles \( \alpha \) and \( \beta \) are supplementary if \( \alpha + \beta = \pi \) and complementary if \( \alpha + \beta = \frac{\pi}{2} \). Finally, we leave it to the reader to show that when using radian measure, two angles \( \alpha \) and \( \beta \) are coterminal if and only if \( \beta = \alpha + 2\pi k \) for some integer \( k \).

**Example 10.1.3.** Graph each of the (oriented) angles below in standard position and classify them according to where their terminal side lies. Find three coterminal angles, at least one of which is positive and one of which is negative.

1. \( \alpha = \frac{\pi}{6} \)
2. \( \beta = -\frac{4\pi}{3} \)
3. \( \gamma = \frac{9\pi}{4} \)
4. \( \phi = -\frac{5\pi}{2} \)

**Solution.**

1. The angle \( \alpha = \frac{\pi}{6} \) is positive, so we draw an angle with its initial side on the positive \( x \)-axis and rotate counter-clockwise \( \frac{\pi/6}{2\pi} = \frac{1}{12} \) of a revolution. Thus \( \alpha \) is a Quadrant I angle. Coterminal angles \( \theta \) are of the form \( \theta = \alpha + 2\pi \cdot k \), for some integer \( k \). To make the arithmetic a bit easier, we note that \( 2\pi = \frac{12\pi}{6} \), thus when \( k = 1 \), we get \( \theta = \frac{\pi}{6} + \frac{12\pi}{6} = \frac{13\pi}{6} \). Substituting \( k = -1 \) gives \( \theta = \frac{\pi}{6} - \frac{12\pi}{6} = -\frac{11\pi}{6} \) and when we let \( k = 2 \), we get \( \theta = \frac{\pi}{6} + \frac{24\pi}{6} = \frac{25\pi}{6} \).

2. Since \( \beta = -\frac{4\pi}{3} \) is negative, we start at the positive \( x \)-axis and rotate clockwise \( \frac{4\pi/3}{2\pi} = \frac{2}{3} \) of a revolution. We find \( \beta \) to be a Quadrant II angle. To find coterminal angles, we proceed as before using \( 2\pi = \frac{6\pi}{3} \), and compute \( \theta = -\frac{4\pi}{3} + \frac{6\pi}{3} \cdot k \) for integer values of \( k \). We obtain \( \frac{2\pi}{3} \), \(-\frac{10\pi}{3}\) and \( \frac{8\pi}{3} \) as coterminal angles.

\(^{13}\)The authors are well aware that we are now identifying radians with real numbers. We will justify this shortly.

\(^{14}\)This, in turn, endows the subtended arcs with an orientation as well. We address this in short order.
10.1 Angles and their Measure

3. Since $\gamma = \frac{9\pi}{4}$ is positive, we rotate counter-clockwise from the positive $x$-axis. One full revolution accounts for $2\pi = \frac{8\pi}{4}$ of the radian measure with $\frac{\pi}{4}$ or $\frac{1}{4}$ of a revolution remaining. We have $\gamma$ as a Quadrant I angle. All angles coterminal with $\gamma$ are of the form $\theta = \frac{9\pi}{4} + \frac{8\pi}{4} \cdot k$, where $k$ is an integer. Working through the arithmetic, we find: $\frac{\pi}{4}$, $-\frac{7\pi}{4}$, and $\frac{17\pi}{4}$.

4. To graph $\phi = -\frac{5\pi}{2}$, we begin our rotation clockwise from the positive $x$-axis. As $2\pi = \frac{4\pi}{2}$, after one full revolution clockwise, we have $\frac{\pi}{2}$ or $\frac{1}{4}$ of a revolution remaining. Since the terminal side of $\phi$ lies on the negative $y$-axis, $\phi$ is a quadrant angle. To find coterminal angles, we compute $\theta = -\frac{5\pi}{2} + \frac{4\pi}{2} \cdot k$ for a few integers $k$ and obtain $-\frac{\pi}{2}$, $\frac{3\pi}{2}$, and $\frac{7\pi}{2}$.

It is worth mentioning that we could have plotted the angles in Example 10.1.3 by first converting them to degree measure and following the procedure set forth in Example 10.1.2. While converting back and forth from degrees and radians is certainly a good skill to have, it is best that you learn to ‘think in radians’ as well as you can ‘think in degrees’. The authors would, however, be
derelict in our duties if we ignored the basic conversion between these systems altogether. Since one revolution counter-clockwise measures $360^\circ$ and the same angle measures $2\pi$ radians, we can use the proportion $\frac{2\pi \text{ radians}}{360^\circ}$, or its reduced equivalent, $\frac{\pi \text{ radians}}{180^\circ}$, as the conversion factor between the two systems. For example, to convert $60^\circ$ to radians we find $60^\circ \left( \frac{\pi \text{ radians}}{180^\circ} \right) = \frac{\pi}{3}$ radians, or simply $\frac{\pi}{3}$. To convert from radian measure back to degrees, we multiply by the ratio $\frac{180^\circ}{\pi \text{ radian}}$. For example, $-\frac{5\pi}{6}$ radians is equal to $\left( -\frac{5\pi}{6} \text{ radians} \right) \left( \frac{180^\circ}{\pi \text{ radians}} \right) = -150^\circ$.\(^{15}\) Of particular interest is the fact that an angle which measures 1 in radian measure is equal to $\frac{180^\circ}{\pi} \approx 57.2958^\circ$.

We summarize these conversions below.

<table>
<thead>
<tr>
<th>Equation 10.1. Degree - Radian Conversion:</th>
</tr>
</thead>
<tbody>
<tr>
<td>• To convert degree measure to radian measure, multiply by $\frac{\pi \text{ radians}}{180^\circ}$</td>
</tr>
<tr>
<td>• To convert radian measure to degree measure, multiply by $\frac{180^\circ}{\pi \text{ radians}}$</td>
</tr>
</tbody>
</table>

In light of Example 10.1.3 and Equation 10.1, the reader may well wonder what the allure of radian measure is. The numbers involved are, admittedly, much more complicated than degree measure. The answer lies in how easily angles in radian measure can be identified with real numbers. Consider the Unit Circle, $x^2 + y^2 = 1$, as drawn below, the angle $\theta$ in standard position and the corresponding arc measuring $s$ units in length. By definition, and the fact that the Unit Circle has radius 1, the radian measure of $\theta$ is $\frac{s}{r} = \frac{s}{1} = s$ so that, once again blurring the distinction between an angle and its measure, we have $\theta = s$. In order to identify real numbers with oriented angles, we make good use of this fact by essentially ‘wrapping’ the real number line around the Unit Circle and associating to each real number $t$ an oriented arc on the Unit Circle with initial point $(1,0)$.

Viewing the vertical line $x = 1$ as another real number line demarcated like the $y$-axis, given a real number $t > 0$, we ‘wrap’ the (vertical) interval $[0, t]$ around the Unit Circle in a counter-clockwise fashion. The resulting arc has a length of $t$ units and therefore the corresponding angle has radian measure equal to $t$. If $t < 0$, we wrap the interval $[t, 0]$ clockwise around the Unit Circle. Since we have defined clockwise rotation as having negative radian measure, the angle determined by this arc has radian measure equal to $t$. If $t = 0$, we are at the point $(1,0)$ on the $x$-axis which corresponds to an angle with radian measure 0. In this way, we identify each real number $t$ with the corresponding angle with radian measure $t$.

\(^{15}\)Note that the negative sign indicates clockwise rotation in both systems, and so it is carried along accordingly.
On the Unit Circle, $\theta = s$. Identifying $t > 0$ with an angle. Identifying $t < 0$ with an angle.

**Example 10.1.4.** Sketch the oriented arc on the Unit Circle corresponding to each of the following real numbers.

1. $t = \frac{3\pi}{4}$
2. $t = -2\pi$
3. $t = -2$
4. $t = 117$

**Solution.**

1. The arc associated with $t = \frac{3\pi}{4}$ is the arc on the Unit Circle which subtends the angle $\frac{3\pi}{4}$ in radian measure. Since $\frac{3\pi}{4}$ is $\frac{3}{8}$ of a revolution, we have an arc which begins at the point $(1, 0)$ proceeds counter-clockwise up to midway through Quadrant II.

2. Since one revolution is $2\pi$ radians, and $t = -2\pi$ is negative, we graph the arc which begins at $(1, 0)$ and proceeds *clockwise* for one full revolution.

3. Like $t = -2\pi$, $t = -2$ is negative, so we begin our arc at $(1, 0)$ and proceed clockwise around the unit circle. Since $\pi \approx 3.14$ and $\frac{\pi}{2} \approx 1.57$, we find that rotating $2$ radians clockwise from the point $(1, 0)$ lands us in Quadrant III. To more accurately place the endpoint, we proceed as we did in Example 10.1.1, successively halving the angle measure until we find $\frac{5\pi}{8} \approx 1.96$ which tells us our arc extends just a bit beyond the quarter mark into Quadrant III.
4. Since 117 is positive, the arc corresponding to \( t = 117 \) begins at \((1, 0)\) and proceeds counter-clockwise. As 117 is much greater than \(2\pi\), we wrap around the Unit Circle several times before finally reaching our endpoint. We approximate \( \frac{117}{2\pi} \) as 18.62 which tells us we complete 18 revolutions counter-clockwise with 0.62, or just shy of \( \frac{5}{8} \) of a revolution to spare. In other words, the terminal side of the angle which measures 117 radians in standard position is just short of being midway through Quadrant III.

\[ t = -2 \quad \text{and} \quad t = 117 \]

10.1.1 APPLICATIONS OF RADIAN MEASURE: CIRCULAR MOTION

Now that we have paired angles with real numbers via radian measure, a whole world of applications awaits us. Our first excursion into this realm comes by way of circular motion. Suppose an object is moving as pictured below along a circular path of radius \( r \) from the point \( P \) to the point \( Q \) in an amount of time \( t \).

Here \( s \) represents a displacement so that \( s > 0 \) means the object is traveling in a counter-clockwise direction and \( s < 0 \) indicates movement in a clockwise direction. Note that with this convention the formula we used to define radian measure, namely \( \theta = \frac{s}{r} \), still holds since a negative value of \( s \) incurred from a clockwise displacement matches the negative we assign to \( \theta \) for a clockwise rotation. In Physics, the average velocity of the object, denoted \( \bar{v} \) and read as ‘\( v \)-bar’, is defined as the average rate of change of the position of the object with respect to time.\(^{16}\) As a result, we

\(^{16}\)See Definition 2.3 in Section 2.1 for a review of this concept.
have \( \vec{v} = \frac{\text{displacement}}{\text{time}} = \frac{s}{t} \). The quantity \( \vec{v} \) has units of \( \frac{\text{length}}{\text{time}} \) and conveys two ideas: the direction in which the object is moving and how fast the position of the object is changing. The contribution of direction in the quantity \( \vec{v} \) is either to make it positive (in the case of counter-clockwise motion) or negative (in the case of clockwise motion), so that the quantity \( |\vec{v}| \) quantifies how fast the object is moving - it is the speed of the object. Measuring \( \theta \) in radians we have \( \theta = \frac{s}{r} \) thus \( s = r\theta \) and

\[
\vec{v} = \frac{s}{t} = \frac{r\theta}{t} = r \cdot \frac{\theta}{t}
\]

The quantity \( \frac{\theta}{t} \) is called the **average angular velocity** of the object. It is denoted by \( \bar{\omega} \) and is read ‘omega-bar’. The quantity \( \bar{\omega} \) is the average rate of change of the angle \( \theta \) with respect to time and thus has units \( \frac{\text{radians}}{\text{time}} \). If \( \bar{\omega} \) is constant throughout the duration of the motion, then it can be shown\(^{17} \) that the average velocities involved, namely \( \vec{v} \) and \( \bar{\omega} \), are the same as their instantaneous counterparts, \( v \) and \( \omega \), respectively. In this case, \( v \) is simply called the ‘velocity’ of the object and is the instantaneous rate of change of the position of the object with respect to time.\(^{18} \) Similarly, \( \omega \) is called the ‘angular velocity’ and is the instantaneous rate of change of the angle with respect to time.

If the path of the object were ‘uncurled’ from a circle to form a line segment, then the velocity of the object on that line segment would be the same as the velocity on the circle. For this reason, the quantity \( v \) is often called the **linear velocity** of the object in order to distinguish it from the angular velocity, \( \omega \). Putting together the ideas of the previous paragraph, we get the following.

**Equation 10.2. Velocity for Circular Motion:** For an object moving on a circular path of radius \( r \) with constant angular velocity \( \omega \), the (linear) velocity of the object is given by \( v = r\omega \).

We need to talk about units here. The units of \( v \) are \( \frac{\text{length}}{\text{time}} \), the units of \( r \) are length only, and the units of \( \omega \) are \( \frac{\text{radians}}{\text{time}} \). Thus the left hand side of the equation \( v = r\omega \) has units \( \frac{\text{length}}{\text{time}} \), whereas the right hand side has units \( \frac{\text{length} \cdot \text{radians}}{\text{time} \cdot \text{time}} = \frac{\text{length} \cdot \text{radians}}{\text{time}} \). The supposed contradiction in units is resolved by remembering that radians are a dimensionless quantity and angles in radian measure are identified with real numbers so that the units \( \frac{\text{length} \cdot \text{radians}}{\text{time}} \) reduce to the units \( \frac{\text{length}}{\text{time}} \). We are long overdue for an example.

**Example 10.1.5.** Assuming that the surface of the Earth is a sphere, any point on the Earth can be thought of as an object traveling on a circle which completes one revolution in (approximately) 24 hours. The path traced out by the point during this 24 hour period is the Latitude of that point. Lakeland Community College is at 41.628° north latitude, and it can be shown\(^{19} \) that the radius of the earth at this Latitude is approximately 2960 miles. Find the linear velocity, in miles per hour, of Lakeland Community College as the world turns.

**Solution.** To use the formula \( v = r\omega \), we first need to compute the angular velocity \( \omega \). The earth makes one revolution in 24 hours, and one revolution is \( 2\pi \) radians, so \( \omega = \frac{2\pi \text{ radians}}{24 \text{ hours}} = \frac{\pi}{12 \text{ hours}} \).

\(^{17}\) You guessed it, using Calculus . . .

\(^{18}\) See the discussion on Page 161 for more details on the idea of an ‘instantaneous’ rate of change.

\(^{19}\) We will discuss how we arrived at this approximation in Example 10.2.6.
where, once again, we are using the fact that radians are real numbers and are dimensionless. (For simplicity’s sake, we are also assuming that we are viewing the rotation of the earth as counterclockwise so $\omega > 0$.) Hence, the linear velocity is

$$v = 2960 \text{ miles} \cdot \frac{\pi}{12 \text{ hours}} \approx 775 \text{ miles/hour}$$

It is worth noting that the quantity $\frac{1 \text{ revolution}}{24 \text{ hours}}$ in Example 10.1.5 is called the **ordinary frequency** of the motion and is usually denoted by the variable $f$. The ordinary frequency is a measure of how often an object makes a complete cycle of the motion. The fact that $\omega = 2\pi f$ suggests that $\omega$ is also a frequency. Indeed, it is called the **angular frequency** of the motion. On a related note, the quantity $T = \frac{1}{f}$ is called the **period** of the motion and is the amount of time it takes for the object to complete one cycle of the motion. In the scenario of Example 10.1.5, the period of the motion is 24 hours, or one day.

The concepts of frequency and period help frame the equation $v = r\omega$ in a new light. That is, if $\omega$ is fixed, points which are farther from the center of rotation need to travel faster to maintain the same angular frequency since they have farther to travel to make one revolution in one period’s time. The distance of the object to the center of rotation is the radius of the circle, $r$, and is the ‘magnification factor’ which relates $\omega$ and $v$. We will have more to say about frequencies and periods in Section 11.1. While we have exhaustively discussed velocities associated with circular motion, we have yet to discuss a more natural question: if an object is moving on a circular path of radius $r$ with a fixed angular velocity (frequency) $\omega$, what is the position of the object at time $t$? The answer to this question is the very heart of Trigonometry and is answered in the next section.
10.2 The Unit Circle: Cosine and Sine

In Section 10.1.1, we introduced circular motion and derived a formula which describes the linear velocity of an object moving on a circular path at a constant angular velocity. One of the goals of this section is to describe the position of such an object. To that end, consider an angle \( \theta \) in standard position and let \( P \) denote the point where the terminal side of \( \theta \) intersects the Unit Circle. By associating the point \( P \) with the angle \( \theta \), we are assigning a position on the Unit Circle to the angle \( \theta \). The \( x \)-coordinate of \( P \) is called the cosine of \( \theta \), written \( \cos(\theta) \), while the \( y \)-coordinate of \( P \) is called the sine of \( \theta \), written \( \sin(\theta) \). The reader is encouraged to verify that these rules used to match an angle with its cosine and sine do, in fact, satisfy the definition of a function. That is, for each angle \( \theta \), there is only one associated value of \( \cos(\theta) \) and only one associated value of \( \sin(\theta) \).

Example 10.2.1. Find the cosine and sine of the following angles.

1. \( \theta = 270^\circ \) 
2. \( \theta = -\pi \) 
3. \( \theta = 45^\circ \) 
4. \( \theta = \pi/6 \) 
5. \( \theta = 60^\circ \)

Solution.

1. To find \( \cos(270^\circ) \) and \( \sin(270^\circ) \), we plot the angle \( \theta = 270^\circ \) in standard position and find the point on the terminal side of \( \theta \) which lies on the Unit Circle. Since \( 270^\circ \) represents \( 3\pi/4 \) of a counter-clockwise revolution, the terminal side of \( \theta \) lies along the negative \( y \)-axis. Hence, the point we seek is \((0, -1)\) so that \( \cos(270^\circ) = 0 \) and \( \sin(270^\circ) = -1 \).

2. The angle \( \theta = -\pi \) represents one half of a clockwise revolution so its terminal side lies on the negative \( x \)-axis. The point on the Unit Circle that lies on the negative \( x \)-axis is \((-1, 0)\) which means \( \cos(-\pi) = -1 \) and \( \sin(-\pi) = 0 \).

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1The etymology of the name ‘sine’ is quite colorful, and the interested reader is invited to research it; the ‘co’ in ‘cosine’ is explained in Section 10.4.
Finding \( \cos(270°) \) and \( \sin(270°) \)

Finding \( \cos(-\pi) \) and \( \sin(-\pi) \)

3. When we sketch \( \theta = 45° \) in standard position, we see that its terminal does not lie along any of the coordinate axes which makes our job of finding the cosine and sine values a bit more difficult. Let \( P(x, y) \) denote the point on the terminal side of \( \theta \) which lies on the Unit Circle. By definition, \( x = \cos(45°) \) and \( y = \sin(45°) \). If we drop a perpendicular line segment from \( P \) to the \( x \)-axis, we obtain a \( 45° - 45° - 90° \) right triangle whose legs have lengths \( x \) and \( y \) units. From Geometry, we get \( y = x \). Since \( P(x, y) \) lies on the Unit Circle, we have \( x^2 + y^2 = 1 \). Substituting \( y = x \) into this equation yields \( 2x^2 = 1 \), or \( x = \pm \sqrt{\frac{1}{2}} = \pm \frac{\sqrt{2}}{2} \). Since \( P(x, y) \) lies in the first quadrant, \( x > 0 \), so \( x = \cos(45°) = \frac{\sqrt{2}}{2} \) and with \( y = x \) we have \( y = \sin(45°) = \frac{\sqrt{2}}{2} \).

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\(^2\text{Can you show this?}\)
4. As before, the terminal side of $\theta = \frac{\pi}{6}$ does not lie on any of the coordinate axes, so we proceed using a triangle approach. Letting $P(x, y)$ denote the point on the terminal side of $\theta$ which lies on the Unit Circle, we drop a perpendicular line segment from $P$ to the $x$-axis to form a $30^\circ - 60^\circ - 90^\circ$ right triangle. After a bit of Geometry\(^3\) we find $y = \frac{1}{2}$ so $\sin \left( \frac{\pi}{6} \right) = \frac{1}{2}$. Since $P(x, y)$ lies on the Unit Circle, we substitute $y = \frac{1}{2}$ into $x^2 + y^2 = 1$ to get $x^2 = \frac{3}{4}$, or $x = \pm \frac{\sqrt{3}}{2}$. Here, $x > 0$ so $x = \cos \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2}$.

5. Plotting $\theta = 60^\circ$ in standard position, we find it is not a quadrantal angle and set about using a triangle approach. Once again, we get a $30^\circ - 60^\circ - 90^\circ$ right triangle and, after the usual computations, find $x = \cos \left( 60^\circ \right) = \frac{1}{2}$ and $y = \sin \left( 60^\circ \right) = \frac{\sqrt{3}}{2}$.

\(^3\)Again, can you show this?
In Example 10.2.1, it was quite easy to find the cosine and sine of the quadrantal angles, but for non-quadrantal angles, the task was much more involved. In these latter cases, we made good use of the fact that the point \( P(x, y) = (\cos(\theta), \sin(\theta)) \) lies on the Unit Circle, \( x^2 + y^2 = 1 \). If we substitute \( x = \cos(\theta) \) and \( y = \sin(\theta) \) into \( x^2 + y^2 = 1 \), we get \((\cos(\theta))^2 + (\sin(\theta))^2 = 1\). An unfortunate convention, which the authors are compelled to perpetuate, is to write \((\cos(\theta))^2\) as \(\cos^2(\theta)\) and \((\sin(\theta))^2\) as \(\sin^2(\theta)\). Rewriting the identity using this convention results in the following theorem, which is without a doubt one of the most important results in Trigonometry.

**Theorem 10.1. The Pythagorean Identity:** For any angle \( \theta \), \(\cos^2(\theta) + \sin^2(\theta) = 1\).

The moniker ‘Pythagorean’ brings to mind the Pythagorean Theorem, from which both the Distance Formula and the equation for a circle are ultimately derived.\(^5\) The word ‘Identity’ reminds us that, regardless of the angle \( \theta \), the equation in Theorem 10.1 is always true. If one of \(\cos(\theta)\) or \(\sin(\theta)\) is known, Theorem 10.1 can be used to determine the other, up to a \((\pm)\) sign. If, in addition, we know where the terminal side of \( \theta \) lies when in standard position, then we can remove the ambiguity of the \((\pm)\) and completely determine the missing value as the next example illustrates.

**Example 10.2.2.** Using the given information about \( \theta \), find the indicated value.

1. If \( \theta \) is a Quadrant II angle with \(\sin(\theta) = \frac{3}{5}\), find \(\cos(\theta)\).
2. If \( \pi < \theta < \frac{3\pi}{2} \) with \(\cos(\theta) = -\frac{\sqrt{5}}{5}\), find \(\sin(\theta)\).
3. If \(\sin(\theta) = 1\), find \(\cos(\theta)\).

**Solution.**

1. When we substitute \(\sin(\theta) = \frac{3}{5}\) into The Pythagorean Identity, \(\cos^2(\theta) + \sin^2(\theta) = 1\), we obtain \(\cos^2(\theta) + \frac{9}{25} = 1\). Solving, we find \(\cos(\theta) = \pm \frac{4}{5}\). Since \(\theta \) is a Quadrant II angle, its terminal side, when plotted in standard position, lies in Quadrant II. Since the \(x\)-coordinates are negative in Quadrant II, \(\cos(\theta)\) is too. Hence, \(\cos(\theta) = -\frac{4}{5}\).

2. Substituting \(\cos(\theta) = -\frac{\sqrt{5}}{5}\) into \(\cos^2(\theta) + \sin^2(\theta) = 1\) gives \(\sin(\theta) = \pm \frac{2}{\sqrt{5}} = \pm \frac{2\sqrt{5}}{5}\). Since we are given that \(\pi < \theta < \frac{3\pi}{2}\), we know \(\theta \) is a Quadrant III angle. Hence both its sine and cosine are negative and we conclude \(\sin(\theta) = -\frac{2\sqrt{5}}{5}\).

3. When we substitute \(\sin(\theta) = 1\) into \(\cos^2(\theta) + \sin^2(\theta) = 1\), we find \(\cos(\theta) = 0\). \(\square\)

Another tool which helps immensely in determining cosines and sines of angles is the symmetry inherent in the Unit Circle. Suppose, for instance, we wish to know the cosine and sine of \(\theta = \frac{5\pi}{6}\). We plot \(\theta\) in standard position below and, as usual, let \(P(x, y)\) denote the point on the terminal side of \(\theta\) which lies on the Unit Circle. Note that the terminal side of \(\theta\) lies \(\frac{\pi}{2}\) radians short of one half revolution. In Example 10.2.1, we determined that \(\cos \left( \frac{\pi}{2} \right) = \frac{\sqrt{3}}{2}\) and \(\sin \left( \frac{\pi}{2} \right) = \frac{1}{2}\). This means

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\(^4\)This is unfortunate from a ‘function notation’ perspective. See Section 10.6.

\(^5\)See Sections 1.1 and 7.2 for details.
that the point on the terminal side of the angle $\frac{\pi}{6}$, when plotted in standard position, is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. From the figure below, it is clear that the point $P(x,y)$ we seek can be obtained by reflecting that point about the y-axis. Hence, $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$.

In the above scenario, the angle $\frac{\pi}{6}$ is called the **reference angle** for the angle $\frac{5\pi}{6}$. In general, for a non-quadrantal angle $\theta$, the reference angle for $\theta$ (usually denoted $\alpha$) is the *acute* angle made between the terminal side of $\theta$ and the x-axis. If $\theta$ is a Quadrant I or IV angle, $\alpha$ is the angle between the terminal side of $\theta$ and the *positive* x-axis; if $\theta$ is a Quadrant II or III angle, $\alpha$ is the angle between the terminal side of $\theta$ and the *negative* x-axis. If we let $P$ denote the point $(\cos(\theta), \sin(\theta))$, then $P$ lies on the Unit Circle. Since the Unit Circle possesses symmetry with respect to the x-axis, y-axis and origin, regardless of where the terminal side of $\theta$ lies, there is a point $Q$ symmetric with $P$ which determines $\theta$’s reference angle, $\alpha$ as seen below.

Reference angle $\alpha$ for a Quadrant I angle

Reference angle $\alpha$ for a Quadrant II angle
We have just outlined the proof of the following theorem.

**Theorem 10.2. Reference Angle Theorem.** Suppose $\alpha$ is the reference angle for $\theta$. Then $\cos(\theta) = \pm \cos(\alpha)$ and $\sin(\theta) = \pm \sin(\alpha)$, where the choice of the (±) depends on the quadrant in which the terminal side of $\theta$ lies.

In light of Theorem 10.2, it pays to know the cosine and sine values for certain common angles. In the table below, we summarize the values which we consider essential and must be memorized.

<table>
<thead>
<tr>
<th>$\theta$ (degrees)</th>
<th>$\theta$ (radians)</th>
<th>$\cos(\theta)$</th>
<th>$\sin(\theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>30°</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>45°</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
</tr>
<tr>
<td>60°</td>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
</tr>
<tr>
<td>90°</td>
<td>$\frac{\pi}{2}$</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

**Example 10.2.3.** Find the cosine and sine of the following angles.

1. $\theta = 225^\circ$  
2. $\theta = \frac{11\pi}{6}$  
3. $\theta = -\frac{5\pi}{4}$  
4. $\theta = \frac{7\pi}{3}$

**Solution.**

1. We begin by plotting $\theta = 225^\circ$ in standard position and find its terminal side overshoots the negative $x$-axis to land in Quadrant III. Hence, we obtain $\theta$’s reference angle $\alpha$ by subtracting: $\alpha = \theta - 180^\circ = 225^\circ - 180^\circ = 45^\circ$. Since $\theta$ is a Quadrant III angle, both $\cos(\theta) < 0$ and
sin(\theta) < 0. The Reference Angle Theorem yields: \( \cos(225^\circ) = -\cos(45^\circ) = -\frac{\sqrt{2}}{2} \) and \( \sin(225^\circ) = -\sin(45^\circ) = -\frac{\sqrt{2}}{2} \).

2. The terminal side of \( \theta = \frac{11\pi}{6} \), when plotted in standard position, lies in Quadrant IV, just shy of the positive \( x \)-axis. To find \( \theta \)'s reference angle \( \alpha \), we subtract: \( \alpha = 2\pi - \theta = 2\pi - \frac{11\pi}{6} = \frac{\pi}{6} \). Since \( \theta \) is a Quadrant IV angle, \( \cos(\theta) > 0 \) and \( \sin(\theta) < 0 \), so the Reference Angle Theorem gives: \( \cos\left(\frac{11\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \) and \( \sin\left(\frac{11\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2} \).

3. To plot \( \theta = -\frac{5\pi}{4} \), we rotate clockwise an angle of \( \frac{5\pi}{4} \) from the positive \( x \)-axis. The terminal side of \( \theta \), therefore, lies in Quadrant II making an angle of \( \alpha = \frac{5\pi}{4} - \pi = \frac{\pi}{4} \) radians with respect to the negative \( x \)-axis. Since \( \theta \) is a Quadrant II angle, the Reference Angle Theorem gives: \( \cos\left(-\frac{5\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \) and \( \sin\left(-\frac{5\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \).

4. Since the angle \( \theta = \frac{7\pi}{3} \) measures more than \( 2\pi = \frac{6\pi}{3} \), we find the terminal side of \( \theta \) by rotating one full revolution followed by an additional \( \alpha = \frac{7\pi}{3} - 2\pi = \frac{\pi}{3} \) radians. Since \( \theta \) and \( \alpha \) are coterminal, \( \cos\left(\frac{7\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \) and \( \sin\left(\frac{7\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \).
The reader may have noticed that when expressed in radian measure, the reference angle for a non-quadrantal angle is easy to spot. Reduced fraction multiples of $\pi$ with a denominator of 6 have $\frac{\pi}{6}$ as a reference angle, those with a denominator of 4 have $\frac{\pi}{4}$ as their reference angle, and those with a denominator of 3 have $\frac{\pi}{3}$ as their reference angle.\(^6\) The Reference Angle Theorem in conjunction with the table of cosine and sine values on Page 722 can be used to generate the following figure, which the authors feel should be committed to memory.

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\(^6\)For once, we have something convenient about using radian measure in contrast to the abstract theoretical nonsense about using them as a ‘natural’ way to match oriented angles with real numbers!
Example 10.2.4. Suppose $\alpha$ is an acute angle with $\cos(\alpha) = \frac{5}{13}$.

1. Find $\sin(\alpha)$ and use this to plot $\alpha$ in standard position.

2. Find the sine and cosine of the following angles:

   (a) $\theta = \pi + \alpha$  
   (b) $\theta = 2\pi - \alpha$  
   (c) $\theta = 3\pi - \alpha$  
   (d) $\theta = \frac{\pi}{2} + \alpha$

Solution.

1. Proceeding as in Example 10.2.2, we substitute $\cos(\alpha) = \frac{5}{13}$ into $\cos^2(\alpha) + \sin^2(\alpha) = 1$ and find $\sin(\alpha) = \pm\frac{12}{13}$. Since $\alpha$ is an acute (and therefore Quadrant I) angle, $\sin(\alpha)$ is positive. Hence, $\sin(\alpha) = \frac{12}{13}$. To plot $\alpha$ in standard position, we begin our rotation on the positive $x$-axis to the ray which contains the point $(\cos(\alpha), \sin(\alpha)) = \left(\frac{5}{13}, \frac{12}{13}\right)$.

2. (a) To find the cosine and sine of $\theta = \pi + \alpha$, we first plot $\theta$ in standard position. We can imagine the sum of the angles $\pi + \alpha$ as a sequence of two rotations: a rotation of $\pi$ radians followed by a rotation of $\alpha$ radians. $^7$ We see that $\alpha$ is the reference angle for $\theta$, so by the Reference Angle Theorem, $\cos(\theta) = \pm \cos(\alpha) = \pm\frac{5}{13}$ and $\sin(\theta) = \pm \sin(\alpha) = \pm\frac{12}{13}$. Since the terminal side of $\theta$ falls in Quadrant III, both $\cos(\theta)$ and $\sin(\theta)$ are negative, hence, $\cos(\theta) = -\frac{5}{13}$ and $\sin(\theta) = -\frac{12}{13}$.

$^7$Since $\pi + \alpha = \alpha + \pi$, $\theta$ may be plotted by reversing the order of rotations given here. You should do this.
(b) Rewriting $\theta = 2\pi - \alpha$ as $\theta = 2\pi + (-\alpha)$, we can plot $\theta$ by visualizing one complete revolution counter-clockwise followed by a clockwise revolution, or ‘backing up,’ of $\alpha$ radians. We see that $\alpha$ is $\theta$’s reference angle, and since $\theta$ is a Quadrant IV angle, the Reference Angle Theorem gives: $\cos(\theta) = \frac{5}{13}$ and $\sin(\theta) = -\frac{12}{13}$.

(c) Taking a cue from the previous problem, we rewrite $\theta = 3\pi - \alpha$ as $\theta = 3\pi + (-\alpha)$. The angle $3\pi$ represents one and a half revolutions counter-clockwise, so that when we ‘back up’ $\alpha$ radians, we end up in Quadrant II. Using the Reference Angle Theorem, we get $\cos(\theta) = -\frac{5}{13}$ and $\sin(\theta) = \frac{12}{13}$. 
(d) To plot $\theta = \frac{\pi}{2} + \alpha$, we first rotate $\frac{\pi}{2}$ radians and follow up with $\alpha$ radians. The reference angle here is not $\alpha$, so the Reference Angle Theorem is not immediately applicable. (It’s important that you see why this is the case. Take a moment to think about this before reading on.) Let $Q(x, y)$ be the point on the terminal side of $\theta$ which lies on the Unit Circle so that $x = \cos(\theta)$ and $y = \sin(\theta)$. Once we graph $\alpha$ in standard position, we use the fact that equal angles subtend equal chords to show that the dotted lines in the figure below are equal. Hence, $x = \cos(\theta) = -\frac{12}{13}$. Similarly, we find $y = \sin(\theta) = \frac{5}{13}$. 

Visualizing $\theta = \frac{\pi}{2} + \alpha$
Our next example asks us to solve some very basic trigonometric equations.\footnote{We will study trigonometric equations more formally in Section 10.7. Enjoy these relatively straightforward exercises while they last!}

**Example 10.2.5.** Find all of the angles which satisfy the given equation.

1. \( \cos(\theta) = \frac{1}{2} \)
2. \( \sin(\theta) = -\frac{1}{2} \)
3. \( \cos(\theta) = 0. \)

**Solution.** Since there is no context in the problem to indicate whether to use degrees or radians, we will default to using radian measure in our answers to each of these problems. This choice will be justified later in the text when we study what is known as Analytic Trigonometry. In those sections to come, radian measure will be the only appropriate angle measure so it is worth the time to become “fluent in radians” now.

1. If \( \cos(\theta) = \frac{1}{2} \), then the terminal side of \( \theta \), when plotted in standard position, intersects the Unit Circle at \( x = \frac{1}{2} \). This means \( \theta \) is a Quadrant I or IV angle with reference angle \( \frac{\pi}{3} \).

   ![Unit Circle with \( \frac{\pi}{3} \) angle](image1)

   One solution in Quadrant I is \( \theta = \frac{\pi}{3} \), and since all other Quadrant I solutions must be coterminal with \( \frac{\pi}{3} \), we find \( \theta = \frac{\pi}{3} + 2\pi k \) for integers \( k \).\footnote{Recall in Section 10.1, two angles in radian measure are coterminal if and only if they differ by an integer multiple of \( 2\pi \). Hence to describe all angles coterminal with a given angle, we add \( 2\pi k \) for integers \( k = 0, \pm 1, \pm 2, \ldots \).} Proceeding similarly for the Quadrant IV case, we find the solution to \( \cos(\theta) = \frac{1}{2} \) here is \( \frac{5\pi}{3} \), so our answer in this Quadrant is \( \theta = \frac{5\pi}{3} + 2\pi k \) for integers \( k \).

2. If \( \sin(\theta) = -\frac{1}{2} \), then when \( \theta \) is plotted in standard position, its terminal side intersects the Unit Circle at \( y = -\frac{1}{2} \). From this, we determine \( \theta \) is a Quadrant III or Quadrant IV angle with reference angle \( \frac{\pi}{6} \).

   ![Unit Circle with \( \frac{\pi}{6} \) angle](image2)
In Quadrant III, one solution is $\frac{7\pi}{6}$, so we capture all Quadrant III solutions by adding integer multiples of $2\pi$: $\theta = \frac{7\pi}{6} + 2\pi k$. In Quadrant IV, one solution is $\frac{11\pi}{6}$ so all the solutions here are of the form $\theta = \frac{11\pi}{6} + 2\pi k$ for integers $k$.

3. The angles with $\cos(\theta) = 0$ are quadrantal angles whose terminal sides, when plotted in standard position, lie along the $y$-axis.

While, technically speaking, $\frac{\pi}{2}$ isn’t a reference angle we can nonetheless use it to find our answers. If we follow the procedure set forth in the previous examples, we find $\theta = \frac{\pi}{2} + 2\pi k$ and $\theta = \frac{3\pi}{2} + 2\pi k$ for integers $k$. While this solution is correct, it can be shortened to $\theta = \frac{\pi}{2} + \pi k$ for integers $k$. (Can you see why this works from the diagram?)

One of the key items to take from Example 10.2.5 is that, in general, solutions to trigonometric equations consist of infinitely many answers. To get a feel for these answers, the reader is encouraged to follow our mantra from Chapter 9 - that is, ‘When in doubt, write it out!’ This is especially important when checking answers to the exercises. For example, another Quadrant IV solution to $\sin(\theta) = -\frac{1}{2}$ is $\theta = -\frac{\pi}{6}$. Hence, the family of Quadrant IV answers to number 2 above could just have easily been written $\theta = -\frac{\pi}{6} + 2\pi k$ for integers $k$. While on the surface, this family may look
different than the stated solution of $\theta = \frac{11\pi}{6} + 2\pi k$ for integers $k$, we leave it to the reader to show they represent the same list of angles.

10.2.1 BEYOND THE UNIT CIRCLE

We began the section with a quest to describe the position of a particle experiencing circular motion. In defining the cosine and sine functions, we assigned to each angle a position on the Unit Circle. In this subsection, we broaden our scope to include circles of radius $r$ centered at the origin. Consider for the moment the acute angle $\theta$ drawn below in standard position. Let $Q(x, y)$ be the point on the terminal side of $\theta$ which lies on the circle $x^2 + y^2 = r^2$, and let $P(x', y')$ be the point on the terminal side of $\theta$ which lies on the Unit Circle. Now consider dropping perpendiculars from $P$ and $Q$ to create two right triangles, $\triangle OPA$ and $\triangle OQB$. These triangles are similar,\(^\text{10}\) thus it follows that $\frac{x}{x'} = \frac{1}{r} = r$, so $x = rx'$ and, similarly, we find $y = ry'$. Since, by definition, $x' = \cos(\theta)$ and $y' = \sin(\theta)$, we get the coordinates of $Q$ to be $x = r \cos(\theta)$ and $y = r \sin(\theta)$. By reflecting these points through the $x$-axis, $y$-axis and origin, we obtain the result for all non-quadrantal angles $\theta$, and we leave it to the reader to verify these formulas hold for the quadrantal angles.

Not only can we describe the coordinates of $Q$ in terms of $\cos(\theta)$ and $\sin(\theta)$ but since the radius of the circle is $r = \sqrt{x^2 + y^2}$, we can also express $\cos(\theta)$ and $\sin(\theta)$ in terms of the coordinates of $Q$. These results are summarized in the following theorem.

**Theorem 10.3.** If $Q(x, y)$ is the point on the terminal side of an angle $\theta$, plotted in standard position, which lies on the circle $x^2 + y^2 = r^2$ then $x = r \cos(\theta)$ and $y = r \sin(\theta)$. Moreover,

$$\cos(\theta) = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \sin(\theta) = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}$$

\(^{10}\)Do you remember why?
Note that in the case of the Unit Circle we have \( r = \sqrt{x^2 + y^2} = 1 \), so Theorem 10.3 reduces to our definitions of \( \cos(\theta) \) and \( \sin(\theta) \).

**Example 10.2.6.**

1. Suppose that the terminal side of an angle \( \theta \), when plotted in standard position, contains the point \( Q(4, -2) \). Find \( \sin(\theta) \) and \( \cos(\theta) \).

2. In Example 10.1.5 in Section 10.1, we approximated the radius of the earth at 41.628° north latitude to be 2960 miles. Justify this approximation if the radius of the Earth at the Equator is approximately 3960 miles.

**Solution.**

1. Using Theorem 10.3 with \( x = 4 \) and \( y = -2 \), we find \( r = \sqrt{(4)^2 + (-2)^2} = 2\sqrt{5} \) so that \( \cos(\theta) = \frac{x}{r} = \frac{4}{2\sqrt{5}} = \frac{2\sqrt{5}}{5} \) and \( \sin(\theta) = \frac{y}{r} = \frac{-2}{2\sqrt{5}} = -\frac{\sqrt{5}}{5} \).

2. Assuming the Earth is a sphere, a cross-section through the poles produces a circle of radius 3960 miles. Viewing the Equator as the \( x \)-axis, the value we seek is the \( x \)-coordinate of the point \( Q(x, y) \) indicated in the figure below.

The terminal side of \( \theta \) contains \( Q(4, -2) \)  

A point on the Earth at 41.628°N  

Using Theorem 10.3, we get \( x = 3960 \cos(41.628°) \). Using a calculator in ‘degree’ mode, we find \( 3960 \cos(41.628°) \approx 2960 \). Hence, the radius of the Earth at North Latitude 41.628° is approximately 2960 miles.
Theorem 10.3 gives us what we need to describe the position of an object traveling in a circular path of radius \( r \) with constant angular velocity \( \omega \). Suppose that at time \( t \), the object has swept out an angle measuring \( \theta \) radians. If we assume that the object is at the point \((r,0)\) when \( t = 0 \), the angle \( \theta \) is in standard position. By definition, \( \omega = \frac{\theta}{t} \) which we rewrite as \( \theta = \omega t \). According to Theorem 10.3, the location of the object \( Q(x, y) \) on the circle is found using the equations \( x = r \cos(\theta) = r \cos(\omega t) \) and \( y = r \sin(\theta) = r \sin(\omega t) \). Hence, at time \( t \), the object is at the point \((r \cos(\omega t), r \sin(\omega t))\). We have just argued the following.

**Equation 10.3.** Suppose an object is traveling in a circular path of radius \( r \) centered at the origin with constant angular velocity \( \omega \). If \( t = 0 \) corresponds to the point \((r,0)\), then the \( x \) and \( y \) coordinates of the object are functions of \( t \) and are given by \( x = r \cos(\omega t) \) and \( y = r \sin(\omega t) \). Here, \( \omega > 0 \) indicates a counter-clockwise direction and \( \omega < 0 \) indicates a clockwise direction.

Example 10.2.7. Suppose we are in the situation of Example 10.1.5. Find the equations of motion of Lakeland Community College as the earth rotates.

Solution. From Example 10.1.5, we take \( r = 2960 \) miles and \( \omega = \frac{\pi}{12 \text{ hours}} \). Hence, the equations of motion are \( x = r \cos(\omega t) = 2960 \cos \left( \frac{\pi}{12} t \right) \) and \( y = r \sin(\omega t) = 2960 \sin \left( \frac{\pi}{12} t \right) \), where \( x \) and \( y \) are measured in miles and \( t \) is measured in hours.

In addition to circular motion, Theorem 10.3 is also the key to developing what is usually called ‘right triangle’ trigonometry.\(^{11}\) As we shall see in the sections to come, many applications in trigonometry involve finding the measures of the angles in, and lengths of the sides of, right triangles. Indeed, we made good use of some properties of right triangles to find the exact values of the cosine and sine of many of the angles in Example 10.2.1, so the following development shouldn’t be that much of a surprise. Consider the generic right triangle below with corresponding acute angle \( \theta \).

\(^{11}\) You may have been exposed to this in High School.
The hypotenuse. We now imagine drawing this triangle in Quadrant I so that the angle \( \theta \) is in standard position with the adjacent side to \( \theta \) lying along the positive \( x \)-axis.

According to the Pythagorean Theorem, \( a^2 + b^2 = c^2 \), so that the point \( P(a, b) \) lies on a circle of radius \( c \). Theorem 10.3 tells us that \( \cos(\theta) = \frac{a}{c} \) and \( \sin(\theta) = \frac{b}{c} \), so we have determined the cosine and sine of \( \theta \) in terms of the lengths of the sides of the right triangle. Thus we have the following theorem.

**Theorem 10.4.** Suppose \( \theta \) is an acute angle residing in a right triangle. If the length of the side adjacent to \( \theta \) is \( a \), the length of the side opposite \( \theta \) is \( b \), and the length of the hypotenuse is \( c \), then \( \cos(\theta) = \frac{a}{c} \) and \( \sin(\theta) = \frac{b}{c} \).

**Example 10.2.8.** Find the measure of the missing angle and the lengths of the missing sides of:

Solution. The first and easiest task is to find the measure of the missing angle. Since the sum of angles of a triangle is 180\(^\circ\), we know that the missing angle has measure 180\(^\circ\) − 30\(^\circ\) − 90\(^\circ\) = 60\(^\circ\). We now proceed to find the lengths of the remaining two sides of the triangle. Let \( c \) denote the length of the hypotenuse of the triangle. By Theorem 10.4, we have \( \cos(30^\circ) = \frac{7}{c} \), or \( c = \frac{7}{\cos(30^\circ)} \).

Since \( \cos(30^\circ) = \frac{\sqrt{3}}{2} \), we have, after the usual fraction gymnastics, \( c = \frac{14\sqrt{3}}{\sqrt{3}} = \frac{14\sqrt{3}}{3} \). At this point, we have two ways to proceed to find the length of the side opposite the 30\(^\circ\) angle, which we’ll denote \( b \). We know the length of the adjacent side is 7 and the length of the hypotenuse is \( \frac{14\sqrt{3}}{3} \), so we
could use the Pythagorean Theorem to find the missing side and solve $(7)^2 + b^2 = \left(\frac{14\sqrt{3}}{3}\right)^2$ for $b$. Alternatively, we could use Theorem 10.4, namely that $\sin(30^\circ) = \frac{b}{c}$. Choosing the latter, we find $b = c \sin(30^\circ) = \frac{14\sqrt{3}}{3} \cdot \frac{1}{2} = \frac{7\sqrt{3}}{3}$. The triangle with all of its data is recorded below.

We close this section by noting that we can easily extend the functions cosine and sine to real numbers by identifying a real number $t$ with the angle $\theta = t$ radians. Using this identification, we define $\cos(t) = \cos(\theta)$ and $\sin(t) = \sin(\theta)$. In practice this means expressions like $\cos(\pi)$ and $\sin(2)$ can be found by regarding the inputs as angles in radian measure or real numbers; the choice is the reader’s. If we trace the identification of real numbers $t$ with angles $\theta$ in radian measure to its roots on page 704, we can spell out this correspondence more precisely. For each real number $t$, we associate an oriented arc $t$ units in length with initial point $(1,0)$ and endpoint $P(\cos(t), \sin(t))$. In the same way we studied polynomial, rational, exponential, and logarithmic functions, we will study the trigonometric functions $f(t) = \cos(t)$ and $g(t) = \sin(t)$. The first order of business is to find the domains and ranges of these functions. Whether we think of identifying the real number $t$ with the angle $\theta = t$ radians, or think of wrapping an oriented arc around the Unit Circle to find coordinates on the Unit Circle, it should be clear that both the cosine and sine functions are defined for all real numbers $t$. In other words, the domain of $f(t) = \cos(t)$ and of $g(t) = \sin(t)$ is $(-\infty, \infty)$. Since $\cos(t)$ and $\sin(t)$ represent $x$- and $y$-coordinates, respectively, of points on the Unit Circle, they both take on all of the values between $-1$ and $1$, inclusive. In other words, the range of $f(t) = \cos(t)$ and of $g(t) = \sin(t)$ is the interval $[-1, 1]$. To summarize:
Theorem 10.5. Domain and Range of the Cosine and Sine Functions:

- The function $f(t) = \cos(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$

- The function $g(t) = \sin(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$

Suppose, as in the Exercises, we are asked to solve an equation such as $\sin(t) = -\frac{1}{2}$. As we have already mentioned, the distinction between $t$ as a real number and as an angle $\theta = t$ radians is often blurred. Indeed, we solve $\sin(t) = -\frac{1}{2}$ in the exact same manner\footnote{Well, to be pedantic, we would be technically using ‘reference numbers’ or ‘reference arcs’ instead of ‘reference angles’ – but the idea is the same.} as we did in Example 10.2.5 number 2. Our solution is only cosmetically different in that the variable used is $t$ rather than $\theta$: $t = \frac{7\pi}{6} + 2\pi k$ or $t = \frac{11\pi}{6} + 2\pi k$ for integers, $k$. We will study the cosine and sine functions in greater detail in Section 10.5. Until then, keep in mind that any properties of cosine and sine developed in the following sections which regard them as functions of angles in radian measure apply equally well if the inputs are regarded as real numbers.
10.3 The Six Circular Functions and Fundamental Identities

In section 10.2, we defined \( \cos(\theta) \) and \( \sin(\theta) \) for angles \( \theta \) using the coordinate values of points on the Unit Circle. As such, these functions earn the moniker circular functions.\(^1\) It turns out that cosine and sine are just two of the six commonly used circular functions which we define below.

**Definition 10.2. The Circular Functions:** Suppose \( \theta \) is an angle plotted in standard position and \( P(x, y) \) is the point on the terminal side of \( \theta \) which lies on the Unit Circle.

- The **cosine** of \( \theta \), denoted \( \cos(\theta) \), is defined by \( \cos(\theta) = x \).
- The **sine** of \( \theta \), denoted \( \sin(\theta) \), is defined by \( \sin(\theta) = y \).
- The **secant** of \( \theta \), denoted \( \sec(\theta) \), is defined by \( \sec(\theta) = \frac{1}{x} \), provided \( x \neq 0 \).
- The **cosecant** of \( \theta \), denoted \( \csc(\theta) \), is defined by \( \csc(\theta) = \frac{1}{y} \), provided \( y \neq 0 \).
- The **tangent** of \( \theta \), denoted \( \tan(\theta) \), is defined by \( \tan(\theta) = \frac{y}{x} \), provided \( x \neq 0 \).
- The **cotangent** of \( \theta \), denoted \( \cot(\theta) \), is defined by \( \cot(\theta) = \frac{x}{y} \), provided \( y \neq 0 \).

While we left the history of the name ‘sine’ as an interesting research project in Section 10.2, the names ‘tangent’ and ‘secant’ can be explained using the diagram below. Consider the acute angle \( \theta \) below in standard position. Let \( P(x, y) \) denote, as usual, the point on the terminal side of \( \theta \) which lies on the Unit Circle and let \( Q(1, y') \) denote the point on the terminal side of \( \theta \) which lies on the vertical line \( x = 1 \).

\(^1\)In Theorem 10.4 we also showed cosine and sine to be functions of an angle residing in a right triangle so we could just as easily call them trigonometric functions. In later sections, you will find that we do indeed use the phrase ‘trigonometric function’ interchangeably with the term ‘circular function’.
The word ‘tangent’ comes from the Latin meaning ‘to touch,’ and for this reason, the line \( x = 1 \) is called a tangent line to the Unit Circle since it intersects, or ‘touches’, the circle at only one point, namely \((1, 0)\). Dropping perpendiculars from \( P \) and \( Q \) creates a pair of similar triangles \( \triangle OPA \) and \( \triangle OQB \). Thus \( \frac{y'}{y} = \frac{x}{1} \) which gives \( y' = \frac{x}{y} = \tan(\theta) \), where this last equality comes from applying Definition 10.2. We have just shown that for acute angles \( \theta \), \( \tan(\theta) \) is the \( y \)-coordinate of the point on the terminal side of \( \theta \) which lies on the line \( x = 1 \) which is tangent to the Unit Circle. Now the word ‘secant’ means ‘to cut’, so a secant line is any line that ‘cuts through’ a circle at two points.\(^2\) The line containing the terminal side of \( \theta \) is a secant line since it intersects the Unit Circle in Quadrants I and III. With the point \( P \) lying on the Unit Circle, the length of the hypotenuse of \( \triangle OPA \) is 1. If we let \( h \) denote the length of the hypotenuse of \( \triangle OQB \), we have from similar triangles that \( \frac{h}{1} = \frac{x}{1} \), or \( h = \frac{x}{y} = \sec(\theta) \). Hence for an acute angle \( \theta \), sec(\( \theta \)) is the length of the line segment which lies on the secant line determined by the terminal side of \( \theta \) and ‘cuts off’ the tangent line \( x = 1 \). Not only do these observations help explain the names of these functions, they serve as the basis for a fundamental inequality needed for Calculus which we’ll explore in the Exercises.

Of the six circular functions, only cosine and sine are defined for all angles. Since \( \cos(\theta) = x \) and \( \sin(\theta) = y \) in Definition 10.2, it is customary to rephrase the remaining four circular functions in terms of cosine and sine. The following theorem is a result of simply replacing \( x \) with \( \cos(\theta) \) and \( y \) with \( \sin(\theta) \) in Definition 10.2.

**Theorem 10.6. Reciprocal and Quotient Identities:**

- \( \sec(\theta) = \frac{1}{\cos(\theta)} \), provided \( \cos(\theta) \neq 0 \); if \( \cos(\theta) = 0 \), \( \sec(\theta) \) is undefined.
- \( \csc(\theta) = \frac{1}{\sin(\theta)} \), provided \( \sin(\theta) \neq 0 \); if \( \sin(\theta) = 0 \), \( \csc(\theta) \) is undefined.
- \( \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \), provided \( \cos(\theta) \neq 0 \); if \( \cos(\theta) = 0 \), \( \tan(\theta) \) is undefined.
- \( \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} \), provided \( \sin(\theta) \neq 0 \); if \( \sin(\theta) = 0 \), \( \cot(\theta) \) is undefined.

It is high time for an example.

**Example 10.3.1.** Find the indicated value, if it exists.

1. \( \sec(60^\circ) \)
2. \( \csc\left(\frac{7\pi}{4}\right) \)
3. \( \cot(3) \)
4. \( \tan(\theta) \), where \( \theta \) is any angle coterminal with \( \frac{3\pi}{2} \).
5. \( \cos(\theta) \), where \( \csc(\theta) = -\sqrt{3} \) and \( \theta \) is a Quadrant IV angle.
6. \( \sin(\theta) \), where \( \tan(\theta) = 3 \) and \( \pi < \theta < \frac{3\pi}{2} \).

\(^2\)Compare this with the definition given in Section 2.1.
Solution.

1. According to Theorem 10.6, \( \sec (60^\circ) = \frac{1}{\cos (60^\circ)} \). Hence, \( \sec (60^\circ) = \frac{1}{(1/2)} = 2 \).

2. Since \( \sin \left( \frac{7\pi}{4} \right) = -\frac{\sqrt{2}}{2} \), \( \csc \left( \frac{7\pi}{4} \right) = \frac{1}{\sin \left( \frac{7\pi}{4} \right)} = \frac{1}{-\sqrt{2}/2} = -\frac{2}{\sqrt{2}} = -\sqrt{2} \).

3. Since \( \theta = 3 \) radians is not one of the ‘common angles’ from Section 10.2, we resort to the calculator for a decimal approximation. Ensuring that the calculator is in radian mode, we find \( \cot (3) = \frac{\cos (3)}{\sin (3)} \approx -7.015 \).

4. If \( \theta \) is coterminal with \( \frac{3\pi}{2} \), then \( \cos (\theta) = \cos \left( \frac{3\pi}{2} \right) = 0 \) and \( \sin (\theta) = \sin \left( \frac{3\pi}{2} \right) = -1 \). Attempting to compute \( \tan (\theta) = \frac{\sin (\theta)}{\cos (\theta)} \) results in \( -\frac{1}{0} \), so \( \tan (\theta) \) is undefined.

5. We are given that \( \csc (\theta) = \frac{1}{\sin (\theta)} = -\sqrt{5} \) so \( \sin (\theta) = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5} \). As we saw in Section 10.2, we can use the Pythagorean Identity, \( \cos^2(\theta) + \sin^2(\theta) = 1 \), to find \( \cos (\theta) \) by knowing \( \sin (\theta) \). Substituting, we get \( \cos^2(\theta) + \left( -\frac{\sqrt{5}}{5} \right)^2 = 1 \), which gives \( \cos^2(\theta) = \frac{4}{5} \), or \( \cos (\theta) = \pm \frac{2\sqrt{5}}{5} \). Since \( \theta \) is a Quadrant IV angle, \( \cos (\theta) > 0 \), so \( \cos (\theta) = \frac{2\sqrt{5}}{5} \).

6. If \( \tan (\theta) = 3 \), then \( \frac{\sin (\theta)}{\cos (\theta)} = 3 \). Be careful - this does NOT mean we can take \( \sin (\theta) = 3 \) and \( \cos (\theta) = 1 \). Instead, from \( \frac{\sin (\theta)}{\cos (\theta)} = 3 \) we get: \( \sin (\theta) = 3 \cos (\theta) \). To relate \( \cos (\theta) \) and \( \sin (\theta) \), we once again employ the Pythagorean Identity, \( \cos^2(\theta) + \sin^2(\theta) = 1 \). Solving \( \sin (\theta) = 3 \cos (\theta) \) for \( \cos (\theta) \), we find \( \cos (\theta) = \frac{1}{3} \sin (\theta) \). Substituting this into the Pythagorean Identity, we find \( \sin^2(\theta) + \left( \frac{1}{3} \sin (\theta) \right)^2 = 1 \). Solving, we get \( \sin^2(\theta) = \frac{9}{10} \) so \( \sin (\theta) = \pm \frac{3\sqrt{10}}{10} \). Since \( \pi < \theta < \frac{3\pi}{2} \), \( \theta \) is a Quadrant III angle. This means \( \sin (\theta) < 0 \), so our final answer is \( \sin (\theta) = -\frac{3\sqrt{10}}{10} \).

While the Reciprocal and Quotient Identities presented in Theorem 10.6 allow us to always reduce problems involving secant, cosecant, tangent and cotangent to problems involving cosine and sine, it is not always convenient to do so.\(^3\) It is worth taking the time to memorize the tangent and cotangent values of the common angles summarized below.

\(^3\)As we shall see shortly, when solving equations involving secant and cosecant, we usually convert back to cosines and sines. However, when solving for tangent or cotangent, we usually stick with what we’re dealt.
Tangent and Cotangent Values of Common Angles

<table>
<thead>
<tr>
<th>θ(degrees)</th>
<th>θ(radians)</th>
<th>tan(θ)</th>
<th>cot(θ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0</td>
<td>0</td>
<td>undefined</td>
</tr>
<tr>
<td>30°</td>
<td>π/6</td>
<td>√3/3</td>
<td>√3</td>
</tr>
<tr>
<td>45°</td>
<td>π/4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>60°</td>
<td>π/3</td>
<td>√3</td>
<td>√3/3</td>
</tr>
<tr>
<td>90°</td>
<td>π/2</td>
<td>undefined</td>
<td>0</td>
</tr>
</tbody>
</table>

Coupling Theorem 10.6 with the Reference Angle Theorem, Theorem 10.2, we get the following.

**Theorem 10.7. Generalized Reference Angle Theorem.** The values of the circular functions of an angle, if they exist, are the same, up to a sign, of the corresponding circular functions of its reference angle. More specifically, if α is the reference angle for θ, then:

\[ \cos(θ) = \pm \cos(α), \ \sin(θ) = \pm \sin(α), \ \sec(θ) = \pm \sec(α), \ \csc(θ) = \pm \csc(α), \ \tan(θ) = \pm \tan(α) \]

and \[ \cot(θ) = \pm \cot(α). \]

The choice of the (±) depends on the quadrant in which the terminal side of θ lies.

We put Theorem 10.7 to good use in the following example.

**Example 10.3.2.** Find all angles which satisfy the given equation.

1. sec(θ) = 2
2. tan(θ) = √3
3. cot(θ) = −1.

**Solution.**

1. To solve sec(θ) = 2, we convert to cosines and get \( \frac{1}{\cos(θ)} = 2 \) or \( \cos(θ) = \frac{1}{2} \). This is the exact same equation we solved in Example 10.2.5, number 1, so we know the answer is: \( θ = \frac{π}{3} + 2πk \) or \( θ = \frac{5π}{3} + 2πk \) for integers k.

2. From the table of common values, we see \( \tan\left(\frac{π}{3}\right) = \sqrt{3} \). According to Theorem 10.7, we know the solutions to \( \tan(θ) = \sqrt{3} \) must, therefore, have a reference angle of \( \frac{π}{3} \). Our next task is to determine in which quadrants the solutions to this equation lie. Since tangent is defined as the ratio \( \frac{y}{x} \) of points \((x, y)\) on the Unit Circle with \( x \neq 0 \), tangent is positive when \( x \) and \( y \) have the same sign (i.e., when they are both positive or both negative.) This happens in Quadrants I and III. In Quadrant I, we get the solutions: \( θ = \frac{π}{3} + 2πk \) for integers k, and for Quadrant III, we get \( θ = \frac{4π}{3} + 2πk \) for integers k. While these descriptions of the solutions are correct, they can be combined into one list as \( θ = \frac{π}{3} + πk \) for integers k. The latter form of the solution is best understood looking at the geometry of the situation in the diagram below.\(^4\)

\(^4\)See Example 10.2.5 number 3 in Section 10.2 for another example of this kind of simplification of the solution.
3. From the table of common values, we see that $\frac{\pi}{4}$ has a cotangent of 1, which means the solutions to $\cot(\theta) = -1$ have a reference angle of $\frac{\pi}{4}$. To find the quadrants in which our solutions lie, we note that $\cot(\theta) = \frac{x}{y}$ for a point $(x, y)$ on the Unit Circle where $y \neq 0$. If $\cot(\theta)$ is negative, then $x$ and $y$ must have different signs (i.e., one positive and one negative.) Hence, our solutions lie in Quadrants II and IV. Our Quadrant II solution is $\theta = \frac{3\pi}{4} + 2\pi k$, and for Quadrant IV, we get $\theta = \frac{7\pi}{4} + 2\pi k$ for integers $k$. Can these lists be combined? Indeed they can - one such way to capture all the solutions is: $\theta = \frac{3\pi}{4} + \pi k$ for integers $k$.

We have already seen the importance of identities in trigonometry. Our next task is to use use the Reciprocal and Quotient Identities found in Theorem 10.6 coupled with the Pythagorean Identity found in Theorem 10.1 to derive new Pythagorean-like identities for the remaining four circular functions. Assuming $\cos(\theta) \neq 0$, we may start with $\cos^2(\theta) + \sin^2(\theta) = 1$ and divide both sides by $\cos^2(\theta)$ to obtain $1 + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$. Using properties of exponents along with the Reciprocal and Quotient Identities, this reduces to $1 + \tan^2(\theta) = \sec^2(\theta)$. If $\sin(\theta) \neq 0$, we can divide both sides of the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ by $\sin^2(\theta)$, apply Theorem 10.6 once again, and obtain $\cot^2(\theta) + 1 = \csc^2(\theta)$. These three Pythagorean Identities are worth memorizing and they, along with some of their other common forms, are summarized in the following theorem.
10.3 The Six Circular Functions and Fundamental Identities

Theorem 10.8. The Pythagorean Identities:

1. \( \cos^2(\theta) + \sin^2(\theta) = 1. \)

   **Common Alternate Forms:**
   - \( 1 - \sin^2(\theta) = \cos^2(\theta) \)
   - \( 1 - \cos^2(\theta) = \sin^2(\theta) \)

2. \( 1 + \tan^2(\theta) = \sec^2(\theta), \) provided \( \cos(\theta) \neq 0. \)

   **Common Alternate Forms:**
   - \( \sec^2(\theta) - \tan^2(\theta) = 1 \)
   - \( \sec^2(\theta) - 1 = \tan^2(\theta) \)

3. \( 1 + \cot^2(\theta) = \csc^2(\theta), \) provided \( \sin(\theta) \neq 0. \)

   **Common Alternate Forms:**
   - \( \csc^2(\theta) - \cot^2(\theta) = 1 \)
   - \( \csc^2(\theta) - 1 = \cot^2(\theta) \)

Trigonometric identities play an important role in not just Trigonometry, but in Calculus as well. We’ll use them in this book to find the values of the circular functions of an angle and solve equations and inequalities. In Calculus, they are needed to simplify otherwise complicated expressions. In the next example, we make good use of the Theorems 10.6 and 10.8.

**Example 10.3.3.** Verify the following identities. Assume that all quantities are defined.

1. \( \frac{1}{\csc(\theta)} = \sin(\theta) \)
2. \( \tan(\theta) = \sin(\theta) \sec(\theta) \)

3. \( (\sec(\theta) - \tan(\theta))(\sec(\theta) + \tan(\theta)) = 1 \)
4. \( \frac{\sec(\theta)}{1 - \tan(\theta)} = \frac{1}{\cos(\theta) - \sin(\theta)} \)

5. \( 6 \sec(\theta) \tan(\theta) = \frac{3}{1 - \sin(\theta)} - \frac{3}{1 + \sin(\theta)} \)
6. \( \frac{\sin(\theta)}{1 - \cos(\theta)} = \frac{1 + \cos(\theta)}{\sin(\theta)} \)

**Solution.** In verifying identities, we typically start with the more complicated side of the equation and use known identities to transform it into the other side of the equation.

1. To verify \( \frac{1}{\csc(\theta)} = \sin(\theta) \), we start with the left side. Using \( \csc(\theta) = \frac{1}{\sin(\theta)} \), we get:
   \[
   \frac{1}{\csc(\theta)} = \frac{1}{\frac{1}{\sin(\theta)}} = \sin(\theta),
   \]
   which is what we were trying to prove.
2. Starting with the right hand side of $\tan(\theta) = \sin(\theta) \sec(\theta)$, we use $\sec(\theta) = \frac{1}{\cos(\theta)}$ and find:

$$\sin(\theta) \sec(\theta) = \sin(\theta) \frac{1}{\cos(\theta)} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta),$$

where the last equality is courtesy of Theorem 10.6.

3. Expanding the left hand side of the equation gives: $(\sec(\theta) - \tan(\theta))(\sec(\theta) + \tan(\theta)) = \sec^2(\theta) - \tan^2(\theta)$. According to Theorem 10.8, $\sec^2(\theta) - \tan^2(\theta) = 1$. Putting it all together,

$$(\sec(\theta) - \tan(\theta))(\sec(\theta) + \tan(\theta)) = \sec^2(\theta) - \tan^2(\theta) = 1.$$

4. While both sides of our last identity contain fractions, the left side affords us more opportunities to use our identities. Substituting $\sec(\theta) = \frac{1}{\cos(\theta)}$ and $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, we get:

$$\frac{\sec(\theta)}{1 - \tan(\theta)} = \frac{1}{\cos(\theta)} \frac{1}{1 - \frac{\sin(\theta)}{\cos(\theta)}} = \frac{1}{1 - \frac{\sin(\theta)}{\cos(\theta)}} \frac{\cos(\theta)}{\cos(\theta)} = \frac{1}{\cos(\theta) - \sin(\theta)},$$

which is exactly what we had set out to show.

5. The right hand side of the equation seems to hold more promise. We get common denominators and add:

$$\frac{3}{1 - \sin(\theta)} - \frac{3}{1 + \sin(\theta)} = \frac{3(1 + \sin(\theta))}{(1 - \sin(\theta))(1 + \sin(\theta))} - \frac{3(1 - \sin(\theta))}{(1 + \sin(\theta))(1 - \sin(\theta))}$$

$$= \frac{3 + 3 \sin(\theta)}{1 - \sin^2(\theta)} - \frac{3 - 3 \sin(\theta)}{1 - \sin^2(\theta)} = \frac{(3 + 3 \sin(\theta)) - (3 - 3 \sin(\theta))}{1 - \sin^2(\theta)} = \frac{6 \sin(\theta)}{1 - \sin^2(\theta)},$$

$^5$Or, to put to another way, earn more partial credit if this were an exam question!
At this point, it is worth pausing to remind ourselves of our goal. We wish to transform this expression into $6 \sec(\theta) \tan(\theta)$. Using a reciprocal and quotient identity, we find $6 \sec(\theta) \tan(\theta) = 6 \left( \frac{1}{\cos(\theta)} \right) \left( \frac{\sin(\theta)}{\cos(\theta)} \right)$. In other words, we need to get cosines in our denominator. Theorem 10.8 tells us $1 - \sin^2(\theta) = \cos^2(\theta)$ so we get:

\[
\frac{3}{1 - \sin(\theta)} - \frac{3}{1 + \sin(\theta)} = \frac{6 \sin(\theta)}{1 - \sin^2(\theta)} = \frac{6 \sin(\theta)}{\cos^2(\theta)}
\]

6. It is debatable which side of the identity is more complicated. One thing which stands out is that the denominator on the left hand side is $1 - \cos(\theta)$, while the numerator of the right hand side is $1 + \cos(\theta)$. This suggests the strategy of starting with the left hand side and multiplying the numerator and denominator by the quantity $1 + \cos(\theta)$:

\[
\frac{\sin(\theta)}{1 - \cos(\theta)} = \frac{\sin(\theta)}{(1 - \cos(\theta))} \cdot \frac{(1 + \cos(\theta))}{(1 + \cos(\theta))} = \frac{\sin(\theta)(1 + \cos(\theta))}{1 - \cos^2(\theta)} = \frac{\sin(\theta)(1 + \cos(\theta))}{\sin^2(\theta)} = \frac{1 + \cos(\theta)}{\sin(\theta)}
\]

In Example 10.3.3 number 6 above, we see that multiplying $1 - \cos(\theta)$ by $1 + \cos(\theta)$ produces a difference of squares that can be simplified to one term using Theorem 10.8. This is exactly the same kind of phenomenon that occurs when we multiply expressions such as $1 - \sqrt{2}$ by $1 + \sqrt{2}$ or $3 - 4i$ by $3 + 4i$. (Can you recall instances from Algebra where we did such things?) For this reason, the quantities $(1 - \cos(\theta))$ and $(1 + \cos(\theta))$ are called ‘Pythagorean Conjugates.’ Below is a list of other common Pythagorean Conjugates.

<table>
<thead>
<tr>
<th>Pythagorean Conjugates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 - \cos(\theta)$ and $1 + \cos(\theta)$: $(1 - \cos(\theta))(1 + \cos(\theta)) = 1 - \cos^2(\theta) = \sin^2(\theta)$</td>
</tr>
<tr>
<td>$1 - \sin(\theta)$ and $1 + \sin(\theta)$: $(1 - \sin(\theta))(1 + \sin(\theta)) = 1 - \sin^2(\theta) = \cos^2(\theta)$</td>
</tr>
<tr>
<td>$\sec(\theta) - 1$ and $\sec(\theta) + 1$: $(\sec(\theta) - 1)(\sec(\theta) + 1) = \sec^2(\theta) - 1 = \tan^2(\theta)$</td>
</tr>
<tr>
<td>$\sec(\theta) - \tan(\theta)$ and $\sec(\theta) + \tan(\theta)$: $(\sec(\theta) - \tan(\theta))(\sec(\theta) + \tan(\theta)) = \sec^2(\theta) - \tan^2(\theta) = 1$</td>
</tr>
<tr>
<td>$\csc(\theta) - 1$ and $\csc(\theta) + 1$: $(\csc(\theta) - 1)(\csc(\theta) + 1) = \csc^2(\theta) - 1 = \cot^2(\theta)$</td>
</tr>
<tr>
<td>$\csc(\theta) - \cot(\theta)$ and $\csc(\theta) + \cot(\theta)$: $(\csc(\theta) - \cot(\theta))(\csc(\theta) + \cot(\theta)) = \csc^2(\theta) - \cot^2(\theta) = 1$</td>
</tr>
</tbody>
</table>
Verifying trigonometric identities requires a healthy mix of tenacity and inspiration. You will need to spend many hours struggling with them just to become proficient in the basics. Like many things in life, there is no short-cut here – there is no complete algorithm for verifying identities. Nevertheless, a summary of some strategies which may be helpful (depending on the situation) is provided below and ample practice is provided for you in the Exercises.

**Strategies for Verifying Identities**

- Try working on the more complicated side of the identity.
- Use the Reciprocal and Quotient Identities in Theorem 10.6 to write functions on one side of the identity in terms of the functions on the other side of the identity. Simplify the resulting complex fractions.
- Add rational expressions with unlike denominators by obtaining common denominators.
- Use the Pythagorean Identities in Theorem 10.8 to ‘exchange’ sines and cosines, secants and tangents, cosecants and cotangents, and simplify sums or differences of squares to one term.
- Multiply numerator and denominator by Pythagorean Conjugates in order to take advantage of the Pythagorean Identities in Theorem 10.8.
- If you find yourself stuck working with one side of the identity, try starting with the other side of the identity and see if you can find a way to bridge the two parts of your work.

### 10.3.1 BEYOND THE UNIT CIRCLE

In Section 10.2, we generalized the cosine and sine functions from coordinates on the Unit Circle to coordinates on circles of radius \( r \). Using Theorem 10.3 in conjunction with Theorem 10.8, we generalize the remaining circular functions in kind.

**Theorem 10.9.** Suppose \( Q(x, y) \) is the point on the terminal side of an angle \( \theta \) (plotted in standard position) which lies on the circle of radius \( r \), \( x^2 + y^2 = r^2 \). Then:

- \( \sec(\theta) = \frac{r}{x} = \frac{\sqrt{x^2 + y^2}}{x} \), provided \( x \neq 0 \).
- \( \csc(\theta) = \frac{r}{y} = \frac{\sqrt{x^2 + y^2}}{y} \), provided \( y \neq 0 \).
- \( \tan(\theta) = \frac{y}{x} \), provided \( x \neq 0 \).
- \( \cot(\theta) = \frac{x}{y} \), provided \( y \neq 0 \).
Example 10.3.4.

1. Suppose the terminal side of $\theta$, when plotted in standard position, contains the point $Q(3, -4)$. Find the values of the six circular functions of $\theta$.

2. Suppose $\theta$ is a Quadrant IV angle with $\cot(\theta) = -4$. Find the values of the five remaining circular functions of $\theta$.

Solution.

1. Since $x = 3$ and $y = -4$, $r = \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$. Theorem 10.9 tells us $\cos(\theta) = \frac{3}{5}$, $\sin(\theta) = -\frac{4}{5}$, $\sec(\theta) = \frac{5}{3}$, $\csc(\theta) = -\frac{5}{4}$, $\tan(\theta) = -\frac{4}{3}$ and $\cot(\theta) = \frac{3}{4}$.

2. In order to use Theorem 10.9, we need to find a point $Q(x, y)$ which lies on the terminal side of $\theta$, when $\theta$ is plotted in standard position. We have that $\cot(\theta) = -4 = \frac{x}{y}$, and since $\theta$ is a Quadrant IV angle, we also know $x > 0$ and $y < 0$. Viewing $-4 = \frac{4}{-1}$, we may choose $x = 4$ and $y = -1$ so that $r = \sqrt{x^2 + y^2} = \sqrt{(4)^2 + (-1)^2} = \sqrt{17}$. Applying Theorem 10.9 once more, we find $\cos(\theta) = \frac{4}{\sqrt{17}}$, $\sin(\theta) = -\frac{1}{\sqrt{17}}$, $\sec(\theta) = \frac{\sqrt{17}}{4}$, $\csc(\theta) = -\frac{\sqrt{17}}{17}$ and $\tan(\theta) = -\frac{1}{4}$.

We may also specialize Theorem 10.9 to the case of acute angles $\theta$ which reside in a right triangle, as visualized below.

![Diagram of a right triangle with sides a, b, and hypotenuse c, and angle $\theta$.]

**Theorem 10.10.** Suppose $\theta$ is an acute angle residing in a right triangle. If the length of the side adjacent to $\theta$ is $a$, the length of the side opposite $\theta$ is $b$, and the length of the hypotenuse is $c$, then

- $\tan(\theta) = \frac{b}{a}$
- $\sec(\theta) = \frac{c}{a}$
- $\csc(\theta) = \frac{c}{b}$
- $\cot(\theta) = \frac{a}{b}$

The following example uses Theorem 10.10 as well as the concept of an ‘angle of inclination.’ The angle of inclination (or angle of elevation) of an object refers to the angle whose initial side is some kind of base-line (say, the ground), and whose terminal side is the line-of-sight to an object above the base-line. This is represented schematically below.

---

6 We may choose *any* values $x$ and $y$ so long as $x > 0$, $y < 0$ and $\frac{x}{y} = -4$. For example, we could choose $x = 8$ and $y = -2$. The fact that all such points lie on the terminal side of $\theta$ is a consequence of the fact that the terminal side of $\theta$ is the portion of the line with slope $-\frac{1}{4}$ which extends from the origin into Quadrant IV.
Example 10.3.5.

1. The angle of inclination from a point on the ground 30 feet away to the top of Lakeland’s Armington Clocktower\(^7\) is 60°. Find the height of the Clocktower to the nearest foot.

2. In order to determine the height of a California Redwood tree, two sightings from the ground, one 200 feet directly behind the other, are made. If the angles of inclination were 45° and 30°, respectively, how tall is the tree to the nearest foot?

Solution.

1. We can represent the problem situation using a right triangle as shown below. If we let \( h \) denote the height of the tower, then Theorem 10.10 gives \( \tan (60^\circ) = \frac{h}{30} \). From this we get \( h = 30 \tan (60^\circ) = 30\sqrt{3} \approx 51.96 \). Hence, the Clocktower is approximately 52 feet tall.

2. Sketching the problem situation below, we find ourselves with two unknowns: the height \( h \) of the tree and the distance \( x \) from the base of the tree to the first observation point.

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\(^7\)Named in honor of Raymond Q. Armington, Lakeland’s Clocktower has been a part of campus since 1972.
10.3 The Six Circular Functions and Fundamental Identities

Finding the height of a California Redwood

Using Theorem 10.10, we get a pair of equations: \( \tan(45^\circ) = \frac{h}{x} \) and \( \tan(30^\circ) = \frac{h}{x+200} \). Since \( \tan(45^\circ) = 1 \), the first equation gives \( \frac{h}{x} = 1 \), or \( x = h \). Substituting this into the second equation gives \( \frac{h}{x+200} = \tan(30^\circ) = \frac{\sqrt{3}}{3} \). Clearing fractions, we get \( 3h = (h + 200)\sqrt{3} \). The result is a linear equation for \( h \), so we proceed to expand the right hand side and gather all the terms involving \( h \) to one side.

\[
\begin{align*}
3h & = (h + 200)\sqrt{3} \\
3h & = h\sqrt{3} + 200\sqrt{3} \\
3h - h\sqrt{3} & = 200\sqrt{3} \\
(3 - \sqrt{3})h & = 200\sqrt{3} \\
h & = \frac{200\sqrt{3}}{3 - \sqrt{3}} \approx 273.20
\end{align*}
\]

Hence, the tree is approximately 273 feet tall.

As we did in Section 10.2.1, we may consider all six circular functions as functions of real numbers. At this stage, there are three equivalent ways to define the functions \( \sec(t) \), \( \csc(t) \), \( \tan(t) \) and \( \cot(t) \) for real numbers \( t \). First, we could go through the formality of the wrapping function on page 704 and define these functions as the appropriate ratios of \( x \) and \( y \) coordinates of points on the Unit Circle; second, we could define them by associating the real number \( t \) with the angle \( \theta = t \) radians so that the value of the trigonometric function of \( t \) coincides with that of \( \theta \); lastly, we could simply define them using the Reciprocal and Quotient Identities as combinations of the functions \( f(t) = \cos(t) \) and \( g(t) = \sin(t) \). Presently, we adopt the last approach. We now set about determining the domains and ranges of the remaining four circular functions. Consider the function \( F(t) = \sec(t) \) defined as \( F(t) = \sec(t) = \frac{1}{\cos(t)} \). We know \( F \) is undefined whenever \( \cos(t) = 0 \). From Example 10.2.5 number 3, we know \( \cos(t) = 0 \) whenever \( t = \frac{\pi}{2} + \pi k \) for integers \( k \). Hence, our domain for \( F(t) = \sec(t) \), in set builder notation is \( \{ t : t \neq \frac{\pi}{2} + \pi k \text{ for integers } k \} \). To get a better understanding what set of real numbers we’re dealing with, it pays to write out and graph this set. Running through a few values of \( k \), we find the domain to be \( \{ t : t \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \ldots \} \). Graphing this set on the number line we get.
Using interval notation to describe this set, we get

\[ \ldots \cup \left( -\frac{5\pi}{2}, -\frac{3\pi}{2} \right) \cup \left( -\frac{3\pi}{2}, -\frac{\pi}{2} \right) \cup \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \cup \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \cup \left( \frac{3\pi}{2}, \frac{5\pi}{2} \right) \cup \ldots \]

This is cumbersome, to say the least! In order to write this in a more compact way, we note that from the set-builder description of the domain, the \( k \)th point excluded from the domain, which we’ll call \( x_k \), can be found by the formula \( x_k = \frac{\pi}{2} + \pi k \). (We are using sequence notation from Chapter 9.)

Getting a common denominator and factoring out the \( \pi \) in the numerator, we get

\[ x_k = \frac{(2k+1)\pi}{4} \]

The domain consists of the intervals determined by successive points \( x_k \): \( (x_k, x_{k+1}) = \left( \frac{(2k+1)\pi}{4}, \frac{(2k+3)\pi}{4} \right) \).

In order to capture all of the intervals in the domain, \( k \) must run through all of the integers, that is, \( k = 0, \pm 1, \pm 2, \ldots \). The way we denote taking the union of infinitely many intervals like this is to use what we call in this text extended interval notation. The domain of \( F(t) = \sec(t) \) can now be written as

\[ \bigcup_{k=-\infty}^{\infty} \left( \frac{(2k+1)\pi}{4}, \frac{(2k+3)\pi}{4} \right) \]

The reader should compare this notation with summation notation introduced in Section 9.2, in particular the notation used to describe geometric series in Theorem 9.2. In the same way the index \( k \) in the series

\[ \sum_{k=1}^{\infty} ar^{k-1} \]

can never equal the upper limit \( \infty \), but rather, ranges through all of the natural numbers, the index \( k \) in the union

\[ \bigcup_{k=-\infty}^{\infty} \left( \frac{(2k+1)\pi}{4}, \frac{(2k+3)\pi}{4} \right) \]

can never actually be \( \infty \) or \(-\infty \), but rather, this conveys the idea that \( k \) ranges through all of the integers. Now that we have painstakingly determined the domain of \( F(t) = \sec(t) \), it is time to discuss the range. Once again, we appeal to the definition \( F(t) = \sec(t) = \frac{1}{\cos(t)} \). The range of \( f(t) = \cos(t) \) is \([-1, 1]\), and since \( F(t) = \sec(t) \) is undefined when \( \cos(t) = 0 \), we split our discussion into two cases: when \( 0 < \cos(t) \leq 1 \) and when \(-1 \leq \cos(t) < 0 \). If \( 0 < \cos(t) \leq 1 \), then we can divide the inequality \( \cos(t) \leq 1 \) by \( \cos(t) \) to obtain \( \sec(t) = \frac{1}{\cos(t)} \geq 1 \). Moreover, using the notation introduced in Section 4.2, we have that as \( \cos(t) \to 0^+ \), \( \sec(t) = \frac{1}{\cos(t)} \approx \frac{1}{\text{very small (+)}} \approx \text{very big (+)} \).

In other words, as \( \cos(t) \to 0^+ \), \( \sec(t) \to \infty \). If, on the other hand, if \(-1 \leq \cos(t) < 0 \), then dividing by \( \cos(t) \) causes a reversal of the inequality so that \( \sec(t) = \frac{1}{\sec(t)} \leq -1 \). In this case, as \( \cos(t) \to 0^- \), \( \sec(t) = \frac{1}{\cos(t)} \approx \frac{1}{\text{very small (−)}} \approx \text{very big (−)} \), so that as \( \cos(t) \to 0^- \), we get \( \sec(t) \to -\infty \). Since
10.3 The Six Circular Functions and Fundamental Identities

The function $f(t) = \cos(t)$ admits all of the values in $[-1, 1]$, the function $F(t) = \sec(t)$ admits all of the values in $(-\infty, -1] \cup [1, \infty)$. Using set-builder notation, the range of $F(t) = \sec(t)$ can be written as $\{u : u \leq -1 \text{ or } u \geq 1\}$, or, more succinctly, $\{u : |u| \geq 1\}$. Similar arguments can be used to determine the domains and ranges of the remaining three circular functions: $\csc(t)$, $\tan(t)$ and $\cot(t)$. The reader is encouraged to do so. (See the Exercises.) For now, we gather these facts into the theorem below.

<table>
<thead>
<tr>
<th>Theorem 10.11. Domains and Ranges of the Circular Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>•</strong> The function $f(t) = \cos(t)$</td>
</tr>
<tr>
<td>- has domain $(-\infty, \infty)$</td>
</tr>
<tr>
<td>- has range $[-1, 1]$</td>
</tr>
<tr>
<td><strong>•</strong> The function $g(t) = \sin(t)$</td>
</tr>
<tr>
<td>- has domain $(-\infty, \infty)$</td>
</tr>
<tr>
<td>- has range $[-1, 1]$</td>
</tr>
<tr>
<td><strong>•</strong> The function $F(t) = \sec(t) = \frac{1}{\cos(t)}$</td>
</tr>
<tr>
<td>- has domain ${t : t \neq \frac{\pi}{2} + \pi k, \text{for integers } k} = \bigcup_{k=-\infty}^{\infty} \left( \frac{(2k + 1)\pi}{2}, \frac{(2k + 3)\pi}{2} \right)$</td>
</tr>
<tr>
<td>- has range ${u :</td>
</tr>
<tr>
<td><strong>•</strong> The function $G(t) = \csc(t) = \frac{1}{\sin(t)}$</td>
</tr>
<tr>
<td>- has domain ${t : t \neq \pi k, \text{for integers } k} = \bigcup_{k=-\infty}^{\infty} (k\pi, (k + 1)\pi)$</td>
</tr>
<tr>
<td>- has range ${u :</td>
</tr>
<tr>
<td><strong>•</strong> The function $J(t) = \tan(t) = \frac{\sin(t)}{\cos(t)}$</td>
</tr>
<tr>
<td>- has domain ${t : t \neq \frac{\pi}{2} + \pi k, \text{for integers } k} = \bigcup_{k=-\infty}^{\infty} \left( \frac{(2k + 1)\pi}{2}, \frac{(2k + 3)\pi}{2} \right)$</td>
</tr>
<tr>
<td>- has range $(-\infty, \infty)$</td>
</tr>
<tr>
<td><strong>•</strong> The function $K(t) = \cot(t) = \frac{\cos(t)}{\sin(t)}$</td>
</tr>
<tr>
<td>- has domain ${t : t \neq \pi k, \text{for integers } k} = \bigcup_{k=-\infty}^{\infty} (k\pi, (k + 1)\pi)$</td>
</tr>
<tr>
<td>- has range $(-\infty, \infty)$</td>
</tr>
</tbody>
</table>

---

8 Using Theorem 2.4 from Section 2.4.
9 Notice we have used the variable ‘u’ as the ‘dummy variable’ to describe the range elements. While there is no mathematical reason to do this (we are describing a set of real numbers, and, as such, could use $t$ again) we choose $u$ to help solidify the idea that these real numbers are the outputs from the inputs, which we have been calling $t$. 
We close this section with a few notes about solving equations which involve the circular functions. First, the discussion on page 735 in Section 10.2.1 concerning solving equations applies to all six circular functions, not just \( f(t) = \cos(t) \) and \( g(t) = \sin(t) \). In particular, to solve the equation \( \cot(t) = -1 \) for real numbers \( t \), we can use the same thought process we used in Example 10.3.2, number 3 to solve \( \cot(\theta) = -1 \) for angles \( \theta \) in radian measure – we just need to remember to write our answers using the variable \( t \) as opposed to \( \theta \). Next, it is critical that you know the domains and ranges of the six circular functions so that you know which equations have no solutions. For example, \( \sec(t) = \frac{1}{2} \) has no solution because \( \frac{1}{2} \) is not in the range of secant. Finally, you will need to review the notions of reference angles and coterminal angles so that you can see why \( \csc(t) = -42 \) has an infinite set of solutions in Quadrant III and another infinite set of solutions in Quadrant IV.
10.4 TRIGONOMETRIC IDENTITIES

In Section 10.3, we saw the utility of the Pythagorean Identities in Theorem 10.8 along with the Quotient and Reciprocal Identities in Theorem 10.6. Not only did these identities help us compute the values of the circular functions for angles, they were also useful in simplifying expressions involving the circular functions. In this section, we introduce several collections of identities which have uses in this course and beyond. Our first set of identities is the ‘Even / Odd’ identities.

**Theorem 10.12. Even / Odd Identities:** For all applicable angles \( \theta \),

\begin{align*}
\cos(-\theta) &= \cos(\theta) \\
\sin(-\theta) &= -\sin(\theta) \\
\sec(-\theta) &= \sec(\theta) \\
\csc(-\theta) &= -\csc(\theta) \\
\tan(-\theta) &= -\tan(\theta) \\
\cot(-\theta) &= -\cot(\theta)
\end{align*}

In light of the Quotient and Reciprocal Identities, Theorem 10.6, it suffices to show \( \cos(-\theta) = \cos(\theta) \) and \( \sin(-\theta) = -\sin(\theta) \). The remaining four circular functions can be expressed in terms of \( \cos(\theta) \) and \( \sin(\theta) \) so the proofs of their Even / Odd Identities are left as exercises. Consider an angle \( \theta \) plotted in standard position. Let \( \theta_0 \) be the angle coterminal with \( \theta \) with \( 0 \leq \theta_0 < 2\pi \). (We can construct the angle \( \theta_0 \) by rotating counter-clockwise from the positive \( x \)-axis to the terminal side of \( \theta \) as pictured below.) Since \( \theta \) and \( \theta_0 \) are coterminal, \( \cos(\theta) = \cos(\theta_0) \) and \( \sin(\theta) = \sin(\theta_0) \).

We now consider the angles \( -\theta \) and \( -\theta_0 \). Since \( \theta \) is coterminal with \( \theta_0 \), there is some integer \( k \) so that \( \theta = \theta_0 + 2\pi \cdot k \). Therefore, \( -\theta = -\theta_0 - 2\pi \cdot k = -\theta_0 + 2\pi \cdot (-k) \). Since \( k \) is an integer, so is \( (-k) \), which means \( -\theta \) is coterminal with \( -\theta_0 \). Hence, \( \cos(-\theta) = \cos(-\theta_0) \) and \( \sin(-\theta) = \sin(-\theta_0) \).

Let \( P \) and \( Q \) denote the points on the terminal sides of \( \theta_0 \) and \( -\theta_0 \), respectively, which lie on the Unit Circle. By definition, the coordinates of \( P \) are \((\cos(\theta_0), \sin(\theta_0))\) and the coordinates of \( Q \) are \((\cos(-\theta_0), \sin(-\theta_0))\). Since \( \theta_0 \) and \( -\theta_0 \) sweep out congruent central sectors of the Unit Circle, it

---

1As mentioned at the end of Section 10.2, properties of the circular functions when thought of as functions of angles in radian measure hold equally well if we view these functions as functions of real numbers. Not surprisingly, the Even / Odd properties of the circular functions are so named because they identify cosine and secant as even functions, while the remaining four circular functions are odd. (See Section 1.6.)
follows that the points $P$ and $Q$ are symmetric about the $x$-axis. Thus, $\cos(-\theta_0) = \cos(\theta_0)$ and $\sin(-\theta_0) = -\sin(\theta_0)$. Since the cosines and sines of $\theta_0$ and $-\theta_0$ are the same as those for $\theta$ and $-\theta$, respectively, we get $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, as required. The Even / Odd Identities are readily demonstrated using any of the ‘common angles’ noted in Section 10.2. Their true utility, however, lies not in computation, but in simplifying expressions involving the circular functions. In fact, our next batch of identities makes heavy use of the Even / Odd Identities.

**Theorem 10.13. Sum and Difference Identities for Cosine:** For all angles $\alpha$ and $\beta$,

- $\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$
- $\cos(\alpha - \beta) = \cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta)$

We first prove the result for differences. As in the proof of the Even / Odd Identities, we can reduce the proof for general angles $\alpha$ and $\beta$ to angles $\alpha_0$ and $\beta_0$, coterminal with $\alpha$ and $\beta$, respectively, each of which measure between 0 and $2\pi$ radians. Since $\alpha$ and $\alpha_0$ are coterminal, as are $\beta$ and $\beta_0$, it follows that $\alpha - \beta$ is coterminal with $\alpha_0 - \beta_0$. Consider the case below where $\alpha_0 \geq \beta_0$.

Since the angles $POQ$ and $AOB$ are congruent, the distance between $P$ and $Q$ is equal to the distance between $A$ and $B$. The distance formula, Equation 1.1, yields

$$\sqrt{(\cos(\alpha) - \cos(\beta))^2 + (\sin(\alpha) - \sin(\beta))^2} = \sqrt{(\cos(\alpha_0 - \beta_0) - 1)^2 + (\sin(\alpha_0 - \beta_0) - 0)^2}$$

Squaring both sides, we expand the left hand side of this equation as

$$(\cos(\alpha) - \cos(\beta))^2 + (\sin(\alpha) - \sin(\beta))^2 = \cos^2(\alpha) - 2 \cos(\alpha) \cos(\beta) + \cos^2(\beta) + \sin^2(\alpha) - 2 \sin(\alpha) \sin(\beta) + \sin^2(\beta)$$

$$= \cos^2(\alpha) + \sin^2(\alpha) + \cos^2(\beta) + \sin^2(\beta) - 2 \cos(\alpha) \cos(\beta) - 2 \sin(\alpha) \sin(\beta)$$

$^2$In the picture we’ve drawn, the triangles $POQ$ and $AOB$ are congruent, which is even better. However, $\alpha_0 - \beta_0$ could be 0 or it could be $\pi$, neither of which makes a triangle. It could also be larger than $\pi$, which makes a triangle, just not the one we’ve drawn. You should think about those three cases.
From the Pythagorean Identities, \( \cos^2(\alpha_o) + \sin^2(\alpha_o) = 1 \) and \( \cos^2(\beta_o) + \sin^2(\beta_o) = 1 \), so

\[
\left( \cos(\alpha_o) - \cos(\beta_o) \right)^2 + \left( \sin(\alpha_o) - \sin(\beta_o) \right)^2 = 2 - 2 \cos(\alpha_o) \cos(\beta_o) - 2 \sin(\alpha_o) \sin(\beta_o)
\]

Turning our attention to the right hand side of our equation, we find

\[
(\cos(\alpha_o - \beta_o) - 1)^2 + (\sin(\alpha_o - \beta_o) - 0)^2 = \cos^2(\alpha_o - \beta_o) - 2 \cos(\alpha_o - \beta_o) + 1 + \sin^2(\alpha_o - \beta_o)
\]

Once again, we simplify \( \cos^2(\alpha_o - \beta_o) + \sin^2(\alpha_o - \beta_o) = 1 \), so that

\[
(\cos(\alpha_o - \beta_o) - 1)^2 + (\sin(\alpha_o - \beta_o) - 0)^2 = 2 - 2 \cos(\alpha_o - \beta_o)
\]

Putting it all together, we get \( 2 - 2 \cos(\alpha_o) \cos(\beta_o) - 2 \sin(\alpha_o) \sin(\beta_o) = 2 - 2 \cos(\alpha_o - \beta_o) \), which simplifies to:

\[
\cos(\alpha_o - \beta_o) = \cos(\alpha_o) \cos(\beta_o) + \sin(\alpha_o) \sin(\beta_o).
\]

For the case where \( \alpha_o \leq \beta_o \), we can apply the above argument to the angle \( \beta_o - \alpha_o \) to obtain the identity \( \cos(\beta_o - \alpha_o) = \cos(\beta_o) \cos(\alpha_o) + \sin(\beta_o) \sin(\alpha_o) \). Applying the Even Identity of cosine, we get \( \cos(\beta_o - \alpha_o) = \cos(-(\alpha_o - \beta_o)) = \cos(\alpha_o - \beta_o) \), and we get the identity in this case, too.

To get the sum identity for cosine, we use the difference formula along with the Even/Odd Identities

\[
\cos(\alpha + \beta) = \cos(\alpha - (-\beta)) = \cos(\alpha) \cos(-\beta) + \sin(\alpha) \sin(-\beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)
\]

We put these newfound identities to good use in the following example.

**Example 10.4.1.**

1. Find the exact value of \( \cos(15^\circ) \).

2. Verify the identity: \( \cos \left( \frac{\pi}{2} - \theta \right) = \sin(\theta) \).

**Solution.**

1. In order to use Theorem 10.13 to find \( \cos(15^\circ) \), we need to write \( 15^\circ \) as a sum or difference of angles whose cosines and sines we know. One way to do so is to write \( 15^\circ = 45^\circ - 30^\circ \).

\[
\cos(15^\circ) = \cos(45^\circ - 30^\circ) = \cos(45^\circ) \cos(30^\circ) + \sin(45^\circ) \sin(30^\circ)
\]

\[
= \left( \frac{\sqrt{2}}{2} \right) \left( \frac{\sqrt{3}}{2} \right) + \left( \frac{\sqrt{2}}{2} \right) \left( \frac{1}{2} \right)
\]

\[
= \frac{\sqrt{6} + \sqrt{2}}{4}
\]
2. In a straightforward application of Theorem 10.13, we find
\[
\cos\left(\frac{\pi}{2} - \theta\right) = \cos\left(\frac{\pi}{2}\right) \cos(\theta) + \sin\left(\frac{\pi}{2}\right) \sin(\theta) \\
= (0) \cos(\theta) + (1) \sin(\theta) \\
= \sin(\theta)
\]

The identity verified in Example 10.4.1, namely, \(\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)\), is the first of the celebrated ‘cofunction’ identities. These identities were first hinted at in Exercise 74 in Section 10.2. From \(\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)\), we get:
\[
\sin\left(\frac{\pi}{2} - \theta\right) = \cos\left(\frac{\pi}{2} - \left[\frac{\pi}{2} - \theta\right]\right) = \cos(\theta),
\]
which says, in words, that the ‘co’sine of an angle is the sine of its ‘co’mplement. Now that these identities have been established for cosine and sine, the remaining circular functions follow suit. The remaining proofs are left as exercises.

**Theorem 10.14. Cofunction Identities:** For all applicable angles \(\theta\),

- \(\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)\)
- \(\sec\left(\frac{\pi}{2} - \theta\right) = \csc(\theta)\)
- \(\tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta)\)
- \(\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)\)
- \(\csc\left(\frac{\pi}{2} - \theta\right) = \sec(\theta)\)
- \(\cot\left(\frac{\pi}{2} - \theta\right) = \tan(\theta)\)

With the Cofunction Identities in place, we are now in the position to derive the sum and difference formulas for sine. To derive the sum formula for sine, we convert to cosines using a cofunction identity, then expand using the difference formula for cosine
\[
\sin(\alpha + \beta) = \cos\left(\frac{\pi}{2} - (\alpha + \beta)\right) \\
= \cos\left(\frac{\pi}{2} - \alpha - \beta\right) \\
= \cos\left(\frac{\pi}{2} - \alpha\right) \cos(\beta) + \sin\left(\frac{\pi}{2} - \alpha\right) \sin(\beta) \\
= \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta)
\]

We can derive the difference formula for sine by rewriting \(\sin(\alpha - \beta)\) as \(\sin(\alpha + (-\beta))\) and using the sum formula and the Even / Odd Identities. Again, we leave the details to the reader.

**Theorem 10.15. Sum and Difference Identities for Sine:** For all angles \(\alpha\) and \(\beta\),

- \(\sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta)\)
- \(\sin(\alpha - \beta) = \sin(\alpha) \cos(\beta) - \cos(\alpha) \sin(\beta)\)
Example 10.4.2.

1. Find the exact value of \( \sin \left( \frac{19\pi}{12} \right) \)

2. If \( \alpha \) is a Quadrant II angle with \( \sin(\alpha) = \frac{5}{13} \), and \( \beta \) is a Quadrant III angle with \( \tan(\beta) = 2 \), find \( \sin(\alpha - \beta) \).

3. Derive a formula for \( \tan(\alpha + \beta) \) in terms of \( \tan(\alpha) \) and \( \tan(\beta) \).

Solution.

1. As in Example 10.4.1, we need to write the angle \( \frac{19\pi}{12} \) as a sum or difference of common angles. The denominator of 12 suggests a combination of angles with denominators 3 and 4. One such combination is \( \frac{19\pi}{12} = \frac{4\pi}{3} + \frac{\pi}{4} \). Applying Theorem 10.15, we get

\[
\sin \left( \frac{19\pi}{12} \right) = \sin \left( \frac{4\pi}{3} + \frac{\pi}{4} \right)
= \sin \left( \frac{4\pi}{3} \right) \cos \left( \frac{\pi}{4} \right) + \cos \left( \frac{4\pi}{3} \right) \sin \left( \frac{\pi}{4} \right)
= \left( -\frac{\sqrt{3}}{2} \right) \left( \frac{\sqrt{2}}{2} \right) + \left( -\frac{1}{2} \right) \left( \frac{\sqrt{2}}{2} \right)
= -\frac{\sqrt{6} - \sqrt{2}}{4}
\]

2. In order to find \( \sin(\alpha - \beta) \) using Theorem 10.15, we need to find \( \cos(\alpha) \) and both \( \cos(\beta) \) and \( \sin(\beta) \). To find \( \cos(\alpha) \), we use the Pythagorean Identity \( \cos^2(\alpha) + \sin^2(\alpha) = 1 \). Since \( \sin(\alpha) = \frac{5}{13} \), we have \( \cos^2(\alpha) + \left( \frac{5}{13} \right)^2 = 1 \), or \( \cos(\alpha) = \pm \frac{12}{13} \). Since \( \alpha \) is a Quadrant II angle, \( \cos(\alpha) = -\frac{12}{13} \). We now set about finding \( \cos(\beta) \) and \( \sin(\beta) \). We have several ways to proceed, but the Pythagorean Identity \( 1 + \tan^2(\beta) = \sec^2(\beta) \) is a quick way to get \( \sec(\beta) \), and hence, \( \cos(\beta) \). With \( \tan(\beta) = 2 \), we get \( 1 + 2^2 = \sec^2(\beta) \) so that \( \sec(\beta) = \pm \sqrt{5} \). Since \( \beta \) is a Quadrant III angle, we choose \( \sec(\beta) = -\sqrt{5} \) so \( \cos(\beta) = \frac{1}{\sec(\beta)} = \frac{1}{-\sqrt{5}} = -\frac{\sqrt{5}}{5} \). We now need to determine \( \sin(\beta) \). We could use The Pythagorean Identity \( \cos^2(\beta) + \sin^2(\beta) = 1 \), but we opt instead to use a quotient identity. From \( \tan(\beta) = \frac{\sin(\beta)}{\cos(\beta)} \), we have \( \sin(\beta) = \tan(\beta) \cos(\beta) \) so we get \( \sin(\beta) = 2 \left( -\frac{\sqrt{5}}{5} \right) = -\frac{2\sqrt{5}}{5} \). We now have all the pieces needed to find \( \sin(\alpha - \beta) \):

\[
\sin(\alpha - \beta) = \sin(\alpha) \cos(\beta) - \cos(\alpha) \sin(\beta)
= \left( \frac{5}{13} \right) \left( -\frac{\sqrt{5}}{5} \right) - \left( -\frac{12}{13} \right) \left( -\frac{2\sqrt{5}}{5} \right)
= -\frac{29\sqrt{5}}{65}
\]
3. We can start expanding $\tan(\alpha + \beta)$ using a quotient identity and our sum formulas

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta)}{\cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)}$$

Since $\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$ and $\tan(\beta) = \frac{\sin(\beta)}{\cos(\beta)}$, it looks as though if we divide both numerator and denominator by $\cos(\alpha) \cos(\beta)$ we will have what we want

$$\tan(\alpha + \beta) = \frac{\sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta)}{\cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)} \frac{1}{\cos(\alpha) \cos(\beta)}$$

$$= \frac{\frac{\sin(\alpha) \cos(\beta)}{\cos(\alpha) \cos(\beta)} + \frac{\cos(\alpha) \sin(\beta)}{\cos(\alpha) \cos(\beta)}}{\frac{\cos(\alpha) \cos(\beta)}{\cos(\alpha) \cos(\beta)} - \frac{\sin(\alpha) \sin(\beta)}{\cos(\alpha) \cos(\beta)}}$$

$$= \frac{\sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta)}{\cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)} \frac{1}{\cos(\alpha) \cos(\beta)}$$

$$= \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha) \tan(\beta)}$$

Naturally, this formula is limited to those cases where all of the tangents are defined.

The formula developed in Exercise 10.4.2 for $\tan(\alpha + \beta)$ can be used to find a formula for $\tan(\alpha - \beta)$ by rewriting the difference as a sum, $\tan(\alpha + (-\beta))$, and the reader is encouraged to fill in the details. Below we summarize all of the sum and difference formulas for cosine, sine and tangent.

**Theorem 10.16. Sum and Difference Identities:** For all applicable angles $\alpha$ and $\beta$,

- $\cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta)$
- $\sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta)$
- $\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha) \tan(\beta)}$

In the statement of Theorem 10.16, we have combined the cases for the sum ‘+’ and difference ‘−’ of angles into one formula. The convention here is that if you want the formula for the sum ‘+’ of
two angles, you use the top sign in the formula; for the difference, ‘−’, use the bottom sign. For example,
\[
\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}
\]
If we specialize the sum formulas in Theorem 10.16 to the case when \( \alpha = \beta \), we obtain the following ‘Double Angle’ Identities.

**Theorem 10.17. Double Angle Identities:** For all applicable angles \( \theta \),

\[
\begin{align*}
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\
2\cos^2(\theta) - 1 &= 1 - 2\sin^2(\theta) \\
\sin(2\theta) &= 2\sin(\theta)\cos(\theta) \\
\tan(2\theta) &= \frac{2\tan(\theta)}{1 - \tan^2(\theta)}
\end{align*}
\]

The three different forms for \( \cos(2\theta) \) can be explained by our ability to ‘exchange’ squares of cosine and sine via the Pythagorean Identity \( \cos^2(\theta) + \sin^2(\theta) = 1 \) and we leave the details to the reader. It is interesting to note that to determine the value of \( \cos(2\theta) \), only one piece of information is required: either \( \cos(\theta) \) or \( \sin(\theta) \). To determine \( \sin(2\theta) \), however, it appears that we must know both \( \sin(\theta) \) and \( \cos(\theta) \). In the next example, we show how we can find \( \sin(2\theta) \) knowing just one piece of information, namely \( \tan(\theta) \).

**Example 10.4.3.**

1. Suppose \( P(-3,4) \) lies on the terminal side of \( \theta \) when \( \theta \) is plotted in standard position. Find \( \cos(2\theta) \) and \( \sin(2\theta) \) and determine the quadrant in which the terminal side of the angle \( 2\theta \) lies when it is plotted in standard position.

2. If \( \sin(\theta) = x \) for \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \), find an expression for \( \sin(2\theta) \) in terms of \( x \).

3. Verify the identity: \( \sin(2\theta) = \frac{2\tan(\theta)}{1 + \tan^2(\theta)} \).

4. Express \( \cos(3\theta) \) as a polynomial in terms of \( \cos(\theta) \).

**Solution.**

1. Using Theorem 10.3 from Section 10.2 with \( x = -3 \) and \( y = 4 \), we find \( r = \sqrt{x^2 + y^2} = 5 \). Hence, \( \cos(\theta) = -\frac{3}{5} \) and \( \sin(\theta) = \frac{4}{5} \). Applying Theorem 10.17, we get \( \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \left( -\frac{3}{5} \right)^2 - \left( \frac{4}{5} \right)^2 = -\frac{7}{25} \), and \( \sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2 \left( \frac{4}{5} \right) \left( -\frac{3}{5} \right) = -\frac{24}{25} \). Since both cosine and sine of \( 2\theta \) are negative, the terminal side of \( 2\theta \), when plotted in standard position, lies in Quadrant III.
2. If your first reaction to ‘\( \sin(\theta) = x \)’ is ‘No it’s not, \( \cos(\theta) = x \)!’ then you have indeed learned something, and we take comfort in that. However, context is everything. Here, ‘\( x \)’ is just a variable - it does not necessarily represent the \( x \)-coordinate of the point on The Unit Circle which lies on the terminal side of \( \theta \), assuming \( \theta \) is drawn in standard position. Here, \( x \) represents the quantity \( \sin(\theta) \), and what we wish to know is how to express \( \sin(2\theta) \) in terms of \( x \). We will see more of this kind of thing in Section 10.6, and, as usual, this is something we need for Calculus. Since \( \sin(2\theta) = 2 \sin(\theta) \cos(\theta) \), we need to write \( \cos(\theta) \) in terms of \( x \) to finish the problem. We substitute \( x = \sin(\theta) \) into the Pythagorean Identity, \( \cos^2(\theta) + \sin^2(\theta) = 1 \), to get \( \cos^2(\theta) + x^2 = 1 \), or \( \cos(\theta) = \pm \sqrt{1 - x^2} \). Since \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \), \( \cos(\theta) \geq 0 \), and thus \( \cos(\theta) = \sqrt{1 - x^2} \). Our final answer is \( \sin(2\theta) = 2 \sin(\theta) \cos(\theta) = 2x\sqrt{1 - x^2} \).

3. We start with the right hand side of the identity and note that \( 1 + \tan^2(\theta) = \sec^2(\theta) \). From this point, we use the Reciprocal and Quotient Identities to rewrite \( \tan(\theta) \) and \( \sec(\theta) \) in terms of \( \cos(\theta) \) and \( \sin(\theta) \):

\[
\frac{2 \tan(\theta)}{1 + \tan^2(\theta)} = \frac{2 \tan(\theta)}{\sec^2(\theta)} = \frac{2 \left( \frac{\sin(\theta)}{\cos(\theta)} \right)}{1} = 2 \left( \frac{\sin(\theta)}{\cos(\theta)} \right) \cos^2(\theta) = 2 \left( \frac{\sin(\theta)}{\cos(\theta)} \right) \cos(\theta) \cos(\theta) = 2 \sin(\theta) \cos(\theta) = \sin(2\theta)
\]

4. In Theorem 10.17, one of the formulas for \( \cos(2\theta) \), namely \( \cos(2\theta) = 2 \cos^2(\theta) - 1 \), expresses \( \cos(2\theta) \) as a polynomial in terms of \( \cos(\theta) \). We are now asked to find such an identity for \( \cos(3\theta) \). Using the sum formula for cosine, we begin with

\[
\cos(3\theta) = \cos(2\theta + \theta) = \cos(2\theta) \cos(\theta) - \sin(2\theta) \sin(\theta)
\]

Our ultimate goal is to express the right hand side in terms of \( \cos(\theta) \) only. We substitute \( \cos(2\theta) = 2 \cos^2(\theta) - 1 \) and \( \sin(2\theta) = 2 \sin(\theta) \cos(\theta) \) which yields

\[
\cos(3\theta) = \cos(2\theta) \cos(\theta) - \sin(2\theta) \sin(\theta) = (2 \cos^2(\theta) - 1) \cos(\theta) - (2 \sin(\theta) \cos(\theta)) \sin(\theta) = 2 \cos^3(\theta) - \cos(\theta) - 2 \sin^2(\theta) \cos(\theta)
\]

Finally, we exchange \( \sin^2(\theta) \) for \( 1 - \cos^2(\theta) \) courtesy of the Pythagorean Identity, and get

\[
\cos(3\theta) = 2 \cos^3(\theta) - \cos(\theta) - 2 \sin^2(\theta) \cos(\theta) = 2 \cos^3(\theta) - \cos(\theta) - 2 (1 - \cos^2(\theta)) \cos(\theta) = 2 \cos^3(\theta) - \cos(\theta) - 2 \cos(\theta) + 2 \cos^3(\theta) = 4 \cos^3(\theta) - 3 \cos(\theta)
\]

and we are done.
In the last problem in Example 10.4.3, we saw how we could rewrite \( \cos(3\theta) \) as sums of powers of \( \cos(\theta) \). In Calculus, we have occasion to do the reverse; that is, reduce the power of cosine and sine. Solving the identity \( \cos(2\theta) = 2\cos^2(\theta) - 1 \) for \( \cos^2(\theta) \) and the identity \( \cos(2\theta) = 1 - 2\sin^2(\theta) \) for \( \sin^2(\theta) \) results in the aptly-named ‘Power Reduction’ formulas below.

**Theorem 10.18. Power Reduction Formulas:** For all angles \( \theta \),

- \( \cos^2(\theta) = \frac{1 + \cos(2\theta)}{2} \)
- \( \sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} \)

**Example 10.4.4.** Rewrite \( \sin^2(\theta) \cos^2(\theta) \) as a sum and difference of cosines to the first power.

**Solution.** We begin with a straightforward application of Theorem 10.18

\[
\sin^2(\theta) \cos^2(\theta) = \left( \frac{1 - \cos(2\theta)}{2} \right) \left( \frac{1 + \cos(2\theta)}{2} \right) \\
= \frac{1}{4} (1 - \cos^2(2\theta)) \\
= \frac{1}{4} - \frac{1}{4} \cos^2(2\theta)
\]

Next, we apply the power reduction formula to \( \cos^2(2\theta) \) to finish the reduction

\[
\sin^2(\theta) \cos^2(\theta) = \frac{1}{4} - \frac{1}{4} \cos^2(2\theta) \\
= \frac{1}{4} - \frac{1}{4} \left( \frac{1 + \cos(2(2\theta))}{2} \right) \\
= \frac{1}{4} - \frac{1}{8} - \frac{1}{8} \cos(4\theta) \\
= \frac{1}{8} - \frac{1}{8} \cos(4\theta)
\]

Another application of the Power Reduction Formulas is the Half Angle Formulas. To start, we apply the Power Reduction Formula to \( \cos^2 \left( \frac{\theta}{2} \right) \)

\[
\cos^2 \left( \frac{\theta}{2} \right) = \frac{1 + \cos \left( \frac{2\theta}{2} \right)}{2} = \frac{1 + \cos(\theta)}{2}.
\]

We can obtain a formula for \( \cos \left( \frac{\theta}{2} \right) \) by extracting square roots. In a similar fashion, we may obtain a half angle formula for sine, and by using a quotient formula, obtain a half angle formula for tangent. We summarize these formulas below.
**Theorem 10.19. Half Angle Formulas:** For all applicable angles $\theta$,

- $\cos \left( \frac{\theta}{2} \right) = \pm \sqrt{\frac{1 + \cos(\theta)}{2}}$
- $\sin \left( \frac{\theta}{2} \right) = \pm \sqrt{\frac{1 - \cos(\theta)}{2}}$
- $\tan \left( \frac{\theta}{2} \right) = \pm \sqrt{\frac{1 - \cos(\theta)}{1 + \cos(\theta)}}$

where the choice of $\pm$ depends on the quadrant in which the terminal side of $\frac{\theta}{2}$ lies.

**Example 10.4.5.**

1. Use a half angle formula to find the exact value of $\cos(15^\circ)$.

2. Suppose $-\pi \leq \theta \leq 0$ with $\cos(\theta) = -\frac{3}{5}$. Find $\sin \left( \frac{\theta}{2} \right)$.

3. Use the identity given in number 3 of Example 10.4.3 to derive the identity

$$\tan \left( \frac{\theta}{2} \right) = \frac{\sin(\theta)}{1 + \cos(\theta)}$$

**Solution.**

1. To use the half angle formula, we note that $15^\circ = \frac{30^\circ}{2}$ and since $15^\circ$ is a Quadrant I angle, its cosine is positive. Thus we have

$$\cos(15^\circ) = \sqrt{\frac{1 + \cos(30^\circ)}{2}} = \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}}$$

$$= \sqrt{1 + \frac{\sqrt{3}}{2}} \cdot \frac{2}{2} = \sqrt{2 + \sqrt{3}} = \frac{\sqrt{2 + \sqrt{3}}}{2}$$

Back in Example 10.4.1, we found $\cos(15^\circ)$ by using the difference formula for cosine. In that case, we determined $\cos(15^\circ) = \frac{\sqrt{6 + \sqrt{2}}}{4}$. The reader is encouraged to prove that these two expressions are equal.

2. If $-\pi \leq \theta \leq 0$, then $-\frac{\pi}{2} \leq \frac{\theta}{2} \leq 0$, which means $\sin \left( \frac{\theta}{2} \right) < 0$. Theorem 10.19 gives

$$\sin \left( \frac{\theta}{2} \right) = -\sqrt{\frac{1 - \cos(\theta)}{2}} = -\sqrt{\frac{1 - (-\frac{3}{5})}{2}}$$

$$= -\sqrt{\frac{1 + \frac{3}{5}}{2}} \cdot \frac{5}{5} = -\sqrt{\frac{8}{10}} = -\frac{2\sqrt{5}}{5}$$
3. Instead of our usual approach to verifying identities, namely starting with one side of the equation and trying to transform it into the other, we will start with the identity we proved in number 3 of Example 10.4.3 and manipulate it into the identity we are asked to prove. The identity we are asked to start with is \( \sin(2\theta) = \frac{2\tan(\theta)}{1 + \tan^2(\theta)} \). If we are to use this to derive an identity for \( \tan\left(\frac{\theta}{2}\right) \), it seems reasonable to proceed by replacing each occurrence of \( \theta \) with \( \frac{\theta}{2} \):

\[
\sin\left(2\left(\frac{\theta}{2}\right)\right) = \frac{2\tan\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)}
\]

We now have the \( \sin(\theta) \) we need, but we somehow need to get a factor of \( 1 + \cos(\theta) \) involved. To get cosines involved, recall that \( 1 + \tan^2\left(\frac{\theta}{2}\right) = \sec^2\left(\frac{\theta}{2}\right) \). We continue to manipulate our given identity by converting secants to cosines and using a power reduction formula:

\[
\sin(\theta) = \frac{2\tan\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)}
\]

Our next batch of identities, the Product to Sum Formulas, are easily verified by expanding each of the right hand sides in accordance with Theorem 10.16 and as you should expect by now we leave the details as exercises. They are of particular use in Calculus, and we list them here for reference.

**Theorem 10.20. Product to Sum Formulas:** For all angles \( \alpha \) and \( \beta \),

\[
\begin{align*}
\cos(\alpha) \cos(\beta) &= \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)] \\
\sin(\alpha) \sin(\beta) &= \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)] \\
\sin(\alpha) \cos(\beta) &= \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)]
\end{align*}
\]

These are also known as the Prosthaphaeresis Formulas and have a rich history. The authors recommend that you conduct some research on them as your schedule allows.
Related to the Product to Sum Formulas are the Sum to Product Formulas, which we will have need of in Section 10.7. These are easily verified using the Product to Sum Formulas, and as such, their proofs are left as exercises.

**Theorem 10.21. Sum to Product Formulas:** For all angles $\alpha$ and $\beta$,

- $\cos(\alpha) + \cos(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$
- $\cos(\alpha) - \cos(\beta) = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$
- $\sin(\alpha) \pm \sin(\beta) = 2 \sin\left(\frac{\alpha \pm \beta}{2}\right) \cos\left(\frac{\alpha \mp \beta}{2}\right)$

**Example 10.4.6.**

1. Write $\cos(2\theta) \cos(6\theta)$ as a sum.
2. Write $\sin(\theta) - \sin(3\theta)$ as a product.

**Solution.**

1. Identifying $\alpha = 2\theta$ and $\beta = 6\theta$, we find

$$
\cos(2\theta) \cos(6\theta) = \frac{1}{2} [\cos(2\theta - 6\theta) + \cos(2\theta + 6\theta)]
$$

$$
= \frac{1}{2} \cos(-4\theta) + \frac{1}{2} \cos(8\theta)
$$

$$
= \frac{1}{2} \cos(4\theta) + \frac{1}{2} \cos(8\theta),
$$

where the last equality is courtesy of the even identity for cosine, $\cos(-4\theta) = \cos(4\theta)$.

2. Identifying $\alpha = \theta$ and $\beta = 3\theta$ yields

$$
\sin(\theta) - \sin(3\theta) = 2 \sin\left(\frac{\theta - 3\theta}{2}\right) \cos\left(\frac{\theta + 3\theta}{2}\right)
$$

$$
= 2 \sin(-\theta) \cos(2\theta)
$$

$$
= -2 \sin(\theta) \cos(2\theta),
$$

where the last equality is courtesy of the odd identity for sine, $\sin(-\theta) = -\sin(\theta)$.

The reader is reminded that all of the identities presented in this section which regard the circular functions as functions of angles (in radian measure) apply equally well to the circular (trigonometric) functions regarded as functions of real numbers. In Exercises 38 - 43 in Section 10.5, we see how some of these identities manifest themselves geometrically as we study the graphs of these functions. In the upcoming Exercises, however, you need to do all of your work analytically without graphs.
10.5 Graphs of the Trigonometric Functions

In this section, we return to our discussion of the circular (trigonometric) functions as functions of real numbers and pick up where we left off in Sections 10.2.1 and 10.3.1. As usual, we begin our study with the functions \( f(t) = \cos(t) \) and \( g(t) = \sin(t) \).

10.5.1 Graphs of the Cosine and Sine Functions

From Theorem 10.5 in Section 10.2.1, we know that the domain of \( f(t) = \cos(t) \) and of \( g(t) = \sin(t) \) is all real numbers, \((-\infty, \infty)\), and the range of both functions is \([-1, 1]\). The Even / Odd Identities in Theorem 10.12 tell us \( \cos(-t) = \cos(t) \) for all real numbers \( t \) and \( \sin(-t) = -\sin(t) \) for all real numbers \( t \). This means \( f(t) = \cos(t) \) is an even function, while \( g(t) = \sin(t) \) is an odd function.\(^1\) Another important property of these functions is that for coterminal angles \( \alpha \) and \( \beta \), \( \cos(\alpha) = \cos(\beta) \) and \( \sin(\alpha) = \sin(\beta) \). Said differently, \( \cos(t + 2\pi k) = \cos(t) \) and \( \sin(t + 2\pi k) = \sin(t) \) for all real numbers \( t \) and any integer \( k \). This last property is given a special name.

**Definition 10.3. Periodic Functions:** A function \( f \) is said to be periodic if there is a real number \( c \) so that \( f(t + c) = f(t) \) for all real numbers \( t \) in the domain of \( f \). The smallest positive number \( p \) for which \( f(t + p) = f(t) \) for all real numbers \( t \) in the domain of \( f \), if it exists, is called the period of \( f \).

We have already seen a family of periodic functions in Section 2.1: the constant functions. However, despite being periodic a constant function has no period. (We’ll leave that odd gem as an exercise for you.) Returning to the circular functions, we see that by Definition 10.3, \( f(t) = \cos(t) \) is periodic, since \( \cos(t + 2\pi k) = \cos(t) \) for any integer \( k \). To determine the period of \( f \), we need to find the smallest real number \( p \) so that \( f(t + p) = f(t) \) for all real numbers \( t \) or, said differently, the smallest positive real number \( p \) such that \( \cos(t + p) = \cos(t) \) for all real numbers \( t \). We know \( \cos(t + 2\pi) = \cos(t) \) for all real numbers \( t \) but the question remains if any smaller real number will do the trick. Suppose \( p > 0 \) and \( \cos(t + p) = \cos(t) \) for all real numbers \( t \). Then, in particular, \( \cos(0 + p) = \cos(0) \) so that \( \cos(p) = 1 \). From this we know \( p \) is a multiple of \( 2\pi \) and, since the smallest positive multiple of \( 2\pi \) is \( 2\pi \) itself, we have the result. Similarly, we can show \( g(t) = \sin(t) \) is also periodic with \( 2\pi \) as its period.\(^2\) Having period \( 2\pi \) essentially means that we can completely understand everything about the functions \( f(t) = \cos(t) \) and \( g(t) = \sin(t) \) by studying one interval of length \( 2\pi \), say \([0, 2\pi]\).\(^3\)

One last property of the functions \( f(t) = \cos(t) \) and \( g(t) = \sin(t) \) is worth pointing out: both of these functions are continuous and smooth. Recall from Section 3.1 that geometrically this means the graphs of the cosine and sine functions have no jumps, gaps, holes in the graph, asymptotes,

---

\(^1\)See section 1.6 for a review of these concepts.

\(^2\)Alternatively, we can use the Cofunction Identities in Theorem 10.14 to show that \( g(t) = \sin(t) \) is periodic with period \( 2\pi \) since \( g(t) = \sin(t) = \cos \left( \frac{\pi}{2} - t \right) = \cos \left( \frac{2\pi}{2} - t \right) \).

\(^3\)Technically, we should study the interval \([0, 2\pi]\), since whatever happens at \( t = 2\pi \) is the same as what happens at \( t = 0 \). As we will see shortly, \( t = 2\pi \) gives us an extra ‘check’ when we go to graph these functions.

\(^4\)In some advanced texts, the interval of choice is \([-\pi, \pi]\).
corners or cusps. As we shall see, the graphs of both \( f(t) = \cos(t) \) and \( g(t) = \sin(t) \) meander nicely and don’t cause any trouble. We summarize these facts in the following theorem.

**Theorem 10.22. Properties of the Cosine and Sine Functions**

- The function \( f(x) = \cos(x) \)
  - has domain \((-\infty, \infty)\)
  - has range \([-1, 1]\)
  - is continuous and smooth
  - is even
  - has period \(2\pi\)

- The function \( g(x) = \sin(x) \)
  - has domain \((-\infty, \infty)\)
  - has range \([-1, 1]\)
  - is continuous and smooth
  - is odd
  - has period \(2\pi\)

In the chart above, we followed the convention established in Section 1.6 and used \( x \) as the independent variable and \( y \) as the dependent variable.\(^5\) This allows us to turn our attention to graphing the cosine and sine functions in the Cartesian Plane. To graph \( y = \cos(x) \), we make a table as we did in Section 1.6 using some of the ‘common values’ of \( x \) in the interval \([0, 2\pi]\). This generates a portion of the cosine graph, which we call the ‘fundamental cycle’ of \( y = \cos(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \cos(x) )</th>
<th>( (x, \cos(x)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>(0, 1)</td>
</tr>
<tr>
<td>( \pi ) 4</td>
<td>( \sqrt{2} ) 2</td>
<td>( (\pi ) 4, ( \sqrt{2} ) 2)</td>
</tr>
<tr>
<td>( \pi ) 2</td>
<td>0</td>
<td>( (\pi ) 2, 0)</td>
</tr>
<tr>
<td>3( \pi ) 4</td>
<td>( -\sqrt{2} ) 2</td>
<td>( (3\pi ) 4, ( -\sqrt{2} ) 2)</td>
</tr>
<tr>
<td>( \pi )</td>
<td>-1</td>
<td>( (\pi ), -1)</td>
</tr>
<tr>
<td>5( \pi ) 4</td>
<td>( -\sqrt{2} ) 2</td>
<td>( (5\pi ) 4, ( -\sqrt{2} ) 2)</td>
</tr>
<tr>
<td>3( \pi ) 2</td>
<td>0</td>
<td>( (3\pi ) 2, 0)</td>
</tr>
<tr>
<td>7( \pi ) 4</td>
<td>( \sqrt{2} ) 2</td>
<td>( (7\pi ) 4, ( \sqrt{2} ) 2)</td>
</tr>
<tr>
<td>2( \pi )</td>
<td>1</td>
<td>( (2\pi ), 1)</td>
</tr>
</tbody>
</table>

A few things about the graph above are worth mentioning. First, this graph represents only part of the graph of \( y = \cos(x) \). To get the entire graph, we imagine ‘copying and pasting’ this graph end to end infinitely in both directions (left and right) on the \( x \)-axis. Secondly, the vertical scale here has been greatly exaggerated for clarity and aesthetics. Below is an accurate-to-scale graph of \( y = \cos(x) \) showing several cycles with the ‘fundamental cycle’ plotted thicker than the others.

\( ^{5} \)The use of \( x \) and \( y \) in this context is not to be confused with the \( x \)- and \( y \)-coordinates of points on the Unit Circle which define cosine and sine. Using the term ‘trigonometric function’ as opposed to ‘circular function’ can help with that, but one could then ask, ‘Hey, where’s the triangle?’
graph of $y = \cos(x)$ is usually described as ‘wavelike’ – indeed, many of the applications involving the cosine and sine functions feature modeling wavelike phenomena.

An accurately scaled graph of $y = \cos(x)$.

We can plot the fundamental cycle of the graph of $y = \sin(x)$ similarly, with similar results.

As with the graph of $y = \cos(x)$, we provide an accurately scaled graph of $y = \sin(x)$ below with the fundamental cycle highlighted.

As with the graph of $y = \cos(x)$, we provide an accurately scaled graph of $y = \sin(x)$ below with the fundamental cycle highlighted.

It is no accident that the graphs of $y = \cos(x)$ and $y = \sin(x)$ are so similar. Using a cofunction identity along with the even property of cosine, we have

$$\sin(x) = \cos\left(\frac{\pi}{2} - x\right) = \cos\left(-\left(x - \frac{\pi}{2}\right)\right) = \cos(x - \frac{\pi}{2})$$

Recalling Section 1.7, we see from this formula that the graph of $y = \sin(x)$ is the result of shifting the graph of $y = \cos(x)$ to the right $\frac{\pi}{2}$ units. A visual inspection confirms this.

Now that we know the basic shapes of the graphs of $y = \cos(x)$ and $y = \sin(x)$, we can use Theorem 1.7 in Section 1.7 to graph more complicated curves. To do so, we need to keep track of
the movement of some key points on the original graphs. We choose to track the values \( x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \) and \( 2\pi \). These ‘quarter marks’ correspond to quadrantal angles, and as such, mark the location of the zeros and the local extrema of these functions over exactly one period. Before we begin our next example, we need to review the concept of the ‘argument’ of a function as first introduced in Section 1.4. For the function \( f(x) = 1 - 5 \cos(2x - \pi) \), the argument of \( f \) is \( x \). We shall have occasion, however, to refer to the argument of the cosine, which in this case is \( 2x - \pi \). Loosely stated, the argument of a trigonometric function is the expression ‘inside’ the function.

**Example 10.5.1.** Graph one cycle of the following functions. State the period of each.

1. \( f(x) = 3 \cos \left( \frac{\pi x - \pi}{2} \right) + 1 \)

2. \( g(x) = \frac{1}{2} \sin(\pi - 2x) + \frac{3}{2} \)

**Solution.**

1. We set the argument of the cosine, \( \frac{\pi x - \pi}{2} \), equal to each of the values: \( 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \) and solve for \( x \). We summarize the results below.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( \frac{\pi x - \pi}{2} = a )</th>
<th>( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \frac{\pi x - \pi}{2} = 0 )</td>
<td>1</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>( \frac{\pi x - \pi}{2} = \frac{\pi}{2} )</td>
<td>2</td>
</tr>
<tr>
<td>( \pi )</td>
<td>( \frac{\pi x - \pi}{2} = \pi )</td>
<td>3</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>( \frac{\pi x - \pi}{2} = \frac{3\pi}{2} )</td>
<td>4</td>
</tr>
<tr>
<td>( 2\pi )</td>
<td>( \frac{\pi x - \pi}{2} = 2\pi )</td>
<td>5</td>
</tr>
</tbody>
</table>

Next, we substitute each of these \( x \) values into \( f(x) = 3 \cos \left( \frac{\pi x - \pi}{2} \right) + 1 \) to determine the corresponding \( y \)-values and connect the dots in a pleasing wavelike fashion.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>((x, f(x)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>(1, 4)</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>(2, 1)</td>
</tr>
<tr>
<td>3</td>
<td>-2</td>
<td>(3, -2)</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>(4, 1)</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>(5, 4)</td>
</tr>
</tbody>
</table>

One cycle is graphed on \([1, 5]\) so the period is the length of that interval which is 4.

2. Proceeding as above, we set the argument of the sine, \( \pi - 2x \), equal to each of our quarter marks and solve for \( x \). 

![Graph of y = f(x)](image-url)
We now find the corresponding \( y \)-values on the graph by substituting each of these \( x \)-values into \( g(x) = \frac{1}{2} \sin(\pi - 2x) + \frac{3}{2} \). Once again, we connect the dots in a wavelike fashion.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( g(x) )</th>
<th>( (x, g(x)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>( \frac{3}{2} )</td>
<td>( \left( \frac{\pi}{2}, \frac{3}{2} \right) )</td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
<td>2</td>
<td>( \left( \frac{\pi}{4}, 2 \right) )</td>
</tr>
<tr>
<td>0</td>
<td>( \frac{3}{2} )</td>
<td>( \left( 0, \frac{3}{2} \right) )</td>
</tr>
<tr>
<td>( -\frac{\pi}{4} )</td>
<td>1</td>
<td>( \left( -\frac{\pi}{4}, 1 \right) )</td>
</tr>
<tr>
<td>( -\frac{\pi}{2} )</td>
<td>( \frac{3}{2} )</td>
<td>( \left( -\frac{\pi}{2}, \frac{3}{2} \right) )</td>
</tr>
</tbody>
</table>

One cycle was graphed on the interval \([ -\frac{\pi}{2}, \frac{\pi}{2} ]\) so the period is \( \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) = \pi \). \( \square \)

The functions in Example 10.5.1 are examples of sinusoids. Roughly speaking, a sinusoid is the result of taking the basic graph of \( f(x) = \cos(x) \) or \( g(x) = \sin(x) \) and performing any of the transformations\(^6\) mentioned in Section 1.7. Sinusoids can be characterized by four properties: period, amplitude, phase shift and vertical shift. We have already discussed period, that is, how long it takes for the sinusoid to complete one cycle. The standard period of both \( f(x) = \cos(x) \) and \( g(x) = \sin(x) \) is \( 2\pi \), but horizontal scalings will change the period of the resulting sinusoid. The amplitude of the sinusoid is a measure of how ‘tall’ the wave is, as indicated in the figure below. The amplitude of the standard cosine and sine functions is 1, but vertical scalings can alter this.

---

\(^6\)We have already seen how the Even/Odd and Cofunction Identities can be used to rewrite \( g(x) = \sin(x) \) as a transformed version of \( f(x) = \cos(x) \), so of course, the reverse is true: \( f(x) = \cos(x) \) can be written as a transformed version of \( g(x) = \sin(x) \). The authors have seen some instances where sinusoids are always converted to cosine functions while in other disciplines, the sinusoids are always written in terms of sine functions. We will discuss the applications of sinusoids in greater detail in Chapter 11. Until then, we will keep our options open.
The phase shift of the sinusoid is the horizontal shift experienced by the fundamental cycle. We have seen that a phase (horizontal) shift of $\frac{\pi}{2}$ to the right takes $f(x) = \cos(x)$ to $g(x) = \sin(x)$ since $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$. As the reader can verify, a phase shift of $\frac{\pi}{2}$ to the left takes $g(x) = \sin(x)$ to $f(x) = \cos(x)$. The vertical shift of a sinusoid is exactly the same as the vertical shifts in Section 1.7. In most contexts, the vertical shift of a sinusoid is assumed to be 0, but we state the more general case below. The following theorem, which is reminiscent of Theorem 1.7 in Section 1.7, shows how to find these four fundamental quantities from the formula of the given sinusoid.

**Theorem 10.23.** For $\omega > 0$, the functions

$$C(x) = A \cos(\omega x + \phi) + B \quad \text{and} \quad S(x) = A \sin(\omega x + \phi) + B$$

- have period $\frac{2\pi}{\omega}$
- have amplitude $|A|$
- have phase shift $-\frac{\phi}{\omega}$
- have vertical shift $B$

We note that in some scientific and engineering circles, the quantity $\phi$ mentioned in Theorem 10.23 is called the phase of the sinusoid. Since our interest in this book is primarily with graphing sinusoids, we focus our attention on the horizontal shift $-\frac{\phi}{\omega}$ induced by $\phi$.

The proof of Theorem 10.23 is a direct application of Theorem 1.7 in Section 1.7 and is left to the reader. The parameter $\omega$, which is stipulated to be positive, is called the (angular) frequency of the sinusoid and is the number of cycles the sinusoid completes over a $2\pi$ interval. We can always ensure $\omega > 0$ using the Even/Odd Identities.\(^7\) We now test out Theorem 10.23 using the functions $f$ and $g$ featured in Example 10.5.1. First, we write $f(x)$ in the form prescribed in Theorem 10.23,

$$f(x) = 3 \cos\left(\frac{\pi x - \pi}{2}\right) + 1 = 3 \cos\left(\frac{\pi}{2} x + \left(-\frac{\pi}{2}\right)\right) + 1,$$

\(^7\)Try using the formulas in Theorem 10.23 applied to $C(x) = \cos(-x + \pi)$ to see why we need $\omega > 0$. 
so that $A = 3$, $\omega = \frac{\pi}{2}$, $\phi = -\frac{\pi}{2}$ and $B = 1$. According to Theorem 10.23, the period of $f$ is \( \frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{2}} = 4 \), the amplitude is $|A| = 3$, the phase shift is $-\frac{\phi}{\omega} = -\frac{-\pi/2}{\pi/2} = 1$ (indicating a shift to the right 1 unit) and the vertical shift is $B = 1$ (indicating a shift up 1 unit.) All of these match with our graph of $y = f(x)$. Moreover, if we start with the basic shape of the cosine graph, shift it 1 unit to the right, 1 unit up, stretch the amplitude to 3 and shrink the period to 4, we will have reconstructed one period of the graph of $y = f(x)$. In other words, instead of tracking the five ‘quarter marks’ through the transformations to plot $y = f(x)$, we can use five other pieces of information: the phase shift, vertical shift, amplitude, period and basic shape of the cosine curve. Turning our attention now to the function $g$ in Example 10.5.1, we first need to use the odd property of the sine function to write it in the form required by Theorem 10.23

$$g(x) = \frac{1}{2} \sin(\pi - 2x) + \frac{3}{2} = \frac{1}{2} \sin(-(2x - \pi)) + \frac{3}{2} = -\frac{1}{2} \sin(2x - \pi) + \frac{3}{2} = -\frac{1}{2} \sin(2x + (-\pi)) + \frac{3}{2}$$

We find $A = -\frac{1}{2}$, $\omega = 2$, $\phi = -\pi$ and $B = \frac{3}{2}$. The period is then $\frac{2\pi}{\omega} = \pi$, the amplitude is $|\frac{-1}{2}| = \frac{1}{2}$, the phase shift is $-\frac{-\pi}{2} = \frac{\pi}{2}$ (indicating a shift right $\frac{\pi}{2}$ units) and the vertical shift is up $\frac{3}{2}$. Note that, in this case, all of the data match our graph of $y = g(x)$ with the exception of the phase shift. Instead of the graph starting at $x = \frac{\pi}{2}$, it ends there. Remember, however, that the graph presented in Example 10.5.1 is only one portion of the graph of $y = g(x)$. Indeed, another complete cycle begins at $x = \frac{\pi}{2}$, and this is the cycle Theorem 10.23 is detecting. The reason for the discrepancy is that, in order to apply Theorem 10.23, we had to rewrite the formula for $g(x)$ using the odd property of the sine function. Note that whether we graph $y = g(x)$ using the ‘quarter marks’ approach or using the Theorem 10.23, we get one complete cycle of the graph, which means we have completely determined the sinusoid.

**Example 10.5.2.** Below is the graph of one complete cycle of a sinusoid $y = f(x)$.

1. Find a cosine function whose graph matches the graph of $y = f(x)$. 

![Graph of y = f(x)](attachment:graph.png)
2. Find a sine function whose graph matches the graph of \( y = f(x) \).

Solution.

1. We fit the data to a function of the form \( C(x) = A\cos(\omega x + \phi) + B \). Since one cycle is graphed over the interval \([-1, 5]\), its period is \( 5 - (-1) = 6 \). According to Theorem 10.23, \( 6 = \frac{2\pi}{\omega} \), so that \( \omega = \frac{\pi}{3} \). Next, we see that the phase shift is \(-1\), so we have \( -\frac{\phi}{\omega} = -1 \), or \( \phi = \omega = \frac{\pi}{3} \). To find the amplitude, note that the range of the sinusoid is \([ -\frac{3}{2}, \frac{5}{2} ] \). As a result, the amplitude \( A = \frac{1}{2} \left( \frac{5}{2} - \left( -\frac{3}{2} \right) \right) = \frac{1}{2} (4) = 2 \). Finally, to determine the vertical shift, we average the endpoints of the range to find \( B = \frac{1}{2} \left( \frac{5}{2} + (-\frac{3}{2}) \right) = \frac{1}{2} (1) = \frac{1}{2} \). Our final answer is \( C(x) = 2 \cos \left( \frac{\pi}{3} x + \frac{\pi}{3} \right) + \frac{1}{2} \).

2. Most of the work to fit the data to a function of the form \( S(x) = A\sin(\omega x + \phi) + B \) is done. The period, amplitude and vertical shift are the same as before with \( \omega = \frac{\pi}{3} \), \( A = 2 \) and \( B = \frac{1}{2} \). The trickier part is finding the phase shift. To that end, we imagine extending the graph of the given sinusoid as in the figure below so that we can identify a cycle beginning at \( \left( \frac{7}{2}, \frac{1}{2} \right) \). Taking the phase shift to be \( \frac{7}{2} \), we get \( -\frac{\phi}{\omega} = \frac{7}{2} \), or \( \phi = -\frac{7}{2} \omega = -\frac{7}{2} \left( \frac{\pi}{3} \right) = -\frac{7\pi}{6} \). Hence, our answer is \( S(x) = 2 \sin \left( \frac{\pi}{3} x - \frac{7\pi}{6} \right) + \frac{1}{2} \).

Note that each of the answers given in Example 10.5.2 is one choice out of many possible answers. For example, when fitting a sine function to the data, we could have chosen to start at \( \left( \frac{1}{2}, \frac{1}{2} \right) \) taking \( A = -2 \). In this case, the phase shift is \( \frac{1}{2} \) so \( \phi = -\frac{\pi}{6} \) for an answer of \( S(x) = -2 \sin \left( \frac{\pi}{3} x - \frac{\pi}{6} \right) + \frac{1}{2} \). Alternatively, we could have extended the graph of \( y = f(x) \) to the left and considered a sine function starting at \( \left( -\frac{3}{2}, \frac{1}{2} \right) \), and so on. Each of these formulas determine the same sinusoid curve and their formulas are all equivalent using identities. Speaking of identities, if we use the sum identity for cosine, we can expand the formula to yield

\[
C(x) = A \cos(\omega x + \phi) + B = A \cos(\omega x) \cos(\phi) - A \sin(\omega x) \sin(\phi) + B.
\]
Similarly, using the sum identity for sine, we get
\[ S(x) = A \sin(\omega x + \phi) + B = A \sin(\omega x) \cos(\phi) + A \cos(\omega x) \sin(\phi) + B. \]

Making these observations allows us to recognize (and graph) functions as sinusoids which, at first glance, don’t appear to fit the forms of either \( C(x) \) or \( S(x) \).

Example 10.5.3. Consider the function \( f(x) = \cos(2x) - \sqrt{3} \sin(2x) \). Find a formula for \( f(x) \):

1. in the form \( C(x) = A \cos(\omega x + \phi) + B \) for \( \omega > 0 \)
2. in the form \( S(x) = A \sin(\omega x + \phi) + B \) for \( \omega > 0 \)

Check your answers analytically using identities and graphically using a calculator.

Solution.

1. The key to this problem is to use the expanded forms of the sinusoid formulas and match up corresponding coefficients. Equating \( f(x) = \cos(2x) - \sqrt{3} \sin(2x) \) with the expanded form of \( C(x) = A \cos(\omega x + \phi) + B \), we get
\[
\cos(2x) - \sqrt{3} \sin(2x) = A \cos(\omega x) \cos(\phi) - A \sin(\omega x) \sin(\phi) + B
\]

It should be clear that we can take \( \omega = 2 \) and \( B = 0 \) to get
\[
\cos(2x) - \sqrt{3} \sin(2x) = A \cos(2x) \cos(\phi) - A \sin(2x) \sin(\phi)
\]

To determine \( A \) and \( \phi \), a bit more work is involved. We get started by equating the coefficients of the trigonometric functions on either side of the equation. On the left hand side, the coefficient of \( \cos(2x) \) is 1, while on the right hand side, it is \( A \cos(\phi) \). Since this equation is to hold for all real numbers, we must have\(^8\) that \( A \cos(\phi) = 1 \). Similarly, we find by equating the coefficients of \( \sin(2x) \) that \( A \sin(\phi) = \sqrt{3} \). What we have here is a system of nonlinear equations! We can temporarily eliminate the dependence on \( \phi \) by using the Pythagorean Identity. We know \( \cos^2(\phi) + \sin^2(\phi) = 1 \), so multiplying this by \( A^2 \) gives \( A^2 \cos^2(\phi) + A^2 \sin^2(\phi) = A^2 \). Since \( A \cos(\phi) = 1 \) and \( A \sin(\phi) = \sqrt{3} \), we get \( A^2 = 1^2 + (\sqrt{3})^2 = 4 \) or \( A = \pm 2 \). Choosing \( A = 2 \), we have \( 2 \cos(\phi) = 1 \) and \( 2 \sin(\phi) = \sqrt{3} \) or, after some rearrangement, \( \cos(\phi) = \frac{1}{2} \) and \( \sin(\phi) = \frac{\sqrt{3}}{2} \). One such angle \( \phi \) which satisfies this criteria is \( \phi = \frac{\pi}{3} \). Hence, one way to write \( f(x) \) as a sinusoid is \( f(x) = 2 \cos(2x + \frac{\pi}{3}) \). We can easily check our answer using the sum formula for cosine
\[
f(x) = 2 \cos \left( 2x + \frac{\pi}{3} \right) \\
= 2 \left[ \cos(2x) \cos \left( \frac{\pi}{3} \right) - \sin(2x) \sin \left( \frac{\pi}{3} \right) \right] \\
= 2 \left[ \cos(2x) \left( \frac{1}{2} \right) - \sin(2x) \left( \frac{\sqrt{3}}{2} \right) \right] \\
= \cos(2x) - \sqrt{3} \sin(2x)
\]

\(^8\)This should remind you of equation coefficients of like powers of \( x \) in Section 8.6.
2. Proceeding as before, we equate $f(x) = \cos(2x) - \sqrt{3}\sin(2x)$ with the expanded form of $S(x) = A\sin(\omega x + \phi) + B$ to get

$$\cos(2x) - \sqrt{3}\sin(2x) = A\sin(\omega x) \cos(\phi) + A\cos(\omega x) \sin(\phi) + B$$

Once again, we may take $\omega = 2$ and $B = 0$ so that

$$\cos(2x) - \sqrt{3}\sin(2x) = A\sin(2x) \cos(\phi) + A\cos(2x) \sin(\phi)$$

We equate\(^9\) the coefficients of $\cos(2x)$ on either side and get $A \sin(\phi) = 1$ and $A \cos(\phi) = -\sqrt{3}$. Using $A^2 \cos^2(\phi) + A^2 \sin^2(\phi) = A^2$ as before, we get $A = \pm 2$, and again we choose $A = 2$. This means $2\sin(\phi) = 1$, or $\sin(\phi) = \frac{1}{2}$, and $2\cos(\phi) = -\sqrt{3}$, which means $\cos(\phi) = -\frac{\sqrt{3}}{2}$. One such angle which meets these criteria is $\phi = \frac{5\pi}{6}$. Hence, we have $f(x) = 2\sin\left(2x + \frac{5\pi}{6}\right)$. Checking our work analytically, we have

$$f(x) = 2 \sin\left(2x + \frac{5\pi}{6}\right)$$

$$= 2 \left[ \sin(2x) \cos\left(\frac{5\pi}{6}\right) + \cos(2x) \sin\left(\frac{5\pi}{6}\right) \right]$$

$$= 2 \left[ \sin(2x) \left(-\frac{\sqrt{3}}{2}\right) + \cos(2x) \left(\frac{1}{2}\right) \right]$$

$$= \cos(2x) - \sqrt{3}\sin(2x)$$

Graphing the three formulas for $f(x)$ result in the identical curve, verifying our analytic work.

![Graph of the functions](image)

It is important to note that in order for the technique presented in Example 10.5.3 to fit a function into one of the forms in Theorem 10.23, the arguments of the cosine and sine function must match. That is, while $f(x) = \cos(2x) - \sqrt{3}\sin(2x)$ is a sinusoid, $g(x) = \cos(2x) - \sqrt{3}\sin(3x)$ is not.\(^{10}\) It is also worth mentioning that, had we chosen $A = -2$ instead of $A = 2$ as we worked through Example 10.5.3, our final answers would have looked different. The reader is encouraged to rework Example 10.5.3 using $A = -2$ to see what these differences are, and then for a challenging exercise, use identities to show that the formulas are all equivalent. The general equations to fit a function of the form $f(x) = a \cos(\omega x) + b \sin(\omega x) + B$ into one of the forms in Theorem 10.23 are explored in Exercise 35.

\(^9\)Be careful here!

\(^{10}\)This graph does, however, exhibit sinusoid-like characteristics! Check it out!
10.5.2 Graphs of the Secant and Cosecant Functions

We now turn our attention to graphing \( y = \sec(x) \). Since \( \sec(x) = \frac{1}{\cos(x)} \), we can use our table of values for the graph of \( y = \cos(x) \) and take reciprocals. We know from Section 10.3.1 that the domain of \( F(x) = \sec(x) \) excludes all odd multiples of \( \frac{\pi}{2} \), and sure enough, we run into trouble at \( x = \frac{\pi}{2} \) and \( x = \frac{3\pi}{2} \) since \( \cos(x) = 0 \) at these values. Using the notation introduced in Section 4.2, we have that as \( x \to \frac{\pi}{2}^- \), \( \cos(x) \to 0^+ \), so \( \sec(x) \to \infty \). (See Section 10.3.1 for a more detailed analysis.) Similarly, we find that as \( x \to \frac{\pi}{2}^+ \), \( \sec(x) \to -\infty \); as \( x \to \frac{3\pi}{2}^- \), \( \sec(x) \to -\infty \); and as \( x \to \frac{3\pi}{2}^+ \), \( \sec(x) \to \infty \). This means we have a pair of vertical asymptotes to the graph of \( y = \sec(x) \), \( x = \frac{\pi}{2} \) and \( x = \frac{3\pi}{2} \). Since \( \cos(x) \) is periodic with period \( 2\pi \), it follows that \( \sec(x) \) is also.

Below we graph a fundamental cycle of \( y = \sec(x) \) along with a more complete graph obtained by the usual ‘copying and pasting.’

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \cos(x) )</th>
<th>( \sec(x) )</th>
<th>( (x, \sec(x)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>(0, 1)</td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
<td>( \sqrt{2}/2 )</td>
<td>( \sqrt{2} )</td>
<td>(( \frac{\pi}{4}, \sqrt{2} ))</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
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<td>undefined</td>
<td></td>
</tr>
<tr>
<td>( \frac{3\pi}{4} )</td>
<td>( -\sqrt{2}/2 )</td>
<td>( -\sqrt{2} )</td>
<td>(( \frac{3\pi}{4}, -\sqrt{2} ))</td>
</tr>
<tr>
<td>( \pi )</td>
<td>-1</td>
<td>-1</td>
<td>(( \pi, -1 ))</td>
</tr>
<tr>
<td>( \frac{5\pi}{4} )</td>
<td>( -\sqrt{2}/2 )</td>
<td>( -\sqrt{2} )</td>
<td>(( \frac{5\pi}{4}, -\sqrt{2} ))</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>0</td>
<td>undefined</td>
<td></td>
</tr>
<tr>
<td>( \frac{7\pi}{4} )</td>
<td>( \sqrt{2}/2 )</td>
<td>( \sqrt{2} )</td>
<td>(( \frac{7\pi}{4}, \sqrt{2} ))</td>
</tr>
<tr>
<td>( 2\pi )</td>
<td>1</td>
<td>1</td>
<td>(( 2\pi, 1 ))</td>
</tr>
</tbody>
</table>

The ‘fundamental cycle’ of \( y = \sec(x) \).

The graph of \( y = \sec(x) \).

\(^{11}\)Provided \( \sec(\alpha) \) and \( \sec(\beta) \) are defined, \( \sec(\alpha) = \sec(\beta) \) if and only if \( \cos(\alpha) = \cos(\beta) \). Hence, \( \sec(x) \) inherits its period from \( \cos(x) \).

\(^{12}\)In Section 10.3.1, we argued the range of \( F(x) = \sec(x) \) is \( (-\infty, -1] \cup [1, \infty) \). We can now see this graphically.
As one would expect, to graph \( y = \csc(x) \) we begin with \( y = \sin(x) \) and take reciprocals of the corresponding \( y \)-values. Here, we encounter issues at \( x = 0 \), \( x = \pi \) and \( x = 2\pi \). Proceeding with the usual analysis, we graph the fundamental cycle of \( y = \csc(x) \) below along with the dotted graph of \( y = \sin(x) \) for reference. Since \( y = \sin(x) \) and \( y = \cos(x) \) are merely phase shifts of each other, so too are \( y = \csc(x) \) and \( y = \sec(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \sin(x) )</th>
<th>( \csc(x) )</th>
<th>( (x, \csc(x)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>undefined</td>
<td></td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \sqrt{2} )</td>
<td>( (\frac{\pi}{4}, \sqrt{2}) )</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>1</td>
<td>1</td>
<td>( (\frac{\pi}{2}, 1) )</td>
</tr>
<tr>
<td>( \frac{3\pi}{4} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \sqrt{2} )</td>
<td>( (\frac{3\pi}{4}, \sqrt{2}) )</td>
</tr>
<tr>
<td>( \pi )</td>
<td>0</td>
<td>undefined</td>
<td></td>
</tr>
<tr>
<td>( \frac{5\pi}{4} )</td>
<td>( -\frac{\sqrt{2}}{2} )</td>
<td>( -\sqrt{2} )</td>
<td>( (\frac{5\pi}{4}, -\sqrt{2}) )</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>-1</td>
<td>-1</td>
<td>( (\frac{3\pi}{2}, -1) )</td>
</tr>
<tr>
<td>( \frac{7\pi}{4} )</td>
<td>( -\frac{\sqrt{2}}{2} )</td>
<td>( -\sqrt{2} )</td>
<td>( (\frac{7\pi}{4}, -\sqrt{2}) )</td>
</tr>
<tr>
<td>( 2\pi )</td>
<td>0</td>
<td>undefined</td>
<td></td>
</tr>
</tbody>
</table>

The ‘fundamental cycle’ of \( y = \csc(x) \).

Once again, our domain and range work in Section 10.3.1 is verified geometrically in the graph of \( y = G(x) = \csc(x) \).

Note that, on the intervals between the vertical asymptotes, both \( F(x) = \sec(x) \) and \( G(x) = \csc(x) \) are continuous and smooth. In other words, they are continuous and smooth on their domains.$^1$ The following theorem summarizes the properties of the secant and cosecant functions. Note that

$^1$Just like the rational functions in Chapter 4 are continuous and smooth on their domains because polynomials are continuous and smooth everywhere, the secant and cosecant functions are continuous and smooth on their domains since the cosine and sine functions are continuous and smooth everywhere.
all of these properties are direct results of them being reciprocals of the cosine and sine functions, respectively.

**Theorem 10.24. Properties of the Secant and Cosecant Functions**

- The function \( F(x) = \sec(x) \)
  - has domain \( \{ x : x \neq \frac{\pi}{2} + \pi k, \text{ } k \text{ is an integer} \} = \bigcup_{k=-\infty}^{\infty} \left( \frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2} \right) \)
  - has range \( \{ y : |y| \geq 1 \} = (-\infty, -1] \cup [1, \infty) \)
  - is continuous and smooth on its domain
  - is even
  - has period \( 2\pi \)

- The function \( G(x) = \csc(x) \)
  - has domain \( \{ x : x \neq \pi k, \text{ } k \text{ is an integer} \} = \bigcup_{k=-\infty}^{\infty} (k\pi, (k+1)\pi) \)
  - has range \( \{ y : |y| \geq 1 \} = (-\infty, -1] \cup [1, \infty) \)
  - is continuous and smooth on its domain
  - is odd
  - has period \( 2\pi \)

In the next example, we discuss graphing more general secant and cosecant curves.

**Example 10.5.4.** Graph one cycle of the following functions. State the period of each.

1. \( f(x) = 1 - 2 \sec(2x) \)
2. \( g(x) = \frac{\csc(\pi - \pi x) - 5}{3} \)

**Solution.**

1. To graph \( y = 1 - 2 \sec(2x) \), we follow the same procedure as in Example 10.5.1. First, we set the argument of secant, \( 2x \), equal to the ‘quarter marks’ \( 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \) and \( 2\pi \) and solve for \( x \).
Next, we substitute these $x$ values into $f(x)$. If $f(x)$ exists, we have a point on the graph; otherwise, we have found a vertical asymptote. In addition to these points and asymptotes, we have graphed the associated cosine curve – in this case $y = 1 - 2 \cos(2x)$ – dotted in the picture below. Since one cycle is graphed over the interval $[0, \pi]$, the period is $\pi - 0 = \pi$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$(x, f(x))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
<td>(0, -1)</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>undefined</td>
<td></td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>3</td>
<td>($\frac{\pi}{2}$, 3)</td>
</tr>
<tr>
<td>$\frac{3\pi}{4}$</td>
<td>undefined</td>
<td></td>
</tr>
<tr>
<td>$\pi$</td>
<td>-1</td>
<td>($\pi$, -1)</td>
</tr>
</tbody>
</table>

One cycle of $y = 1 - 2 \sec(2x)$.

2. Proceeding as before, we set the argument of cosecant in $g(x) = \csc(\pi - \pi x) - \frac{5}{3}$ equal to the quarter marks and solve for $x$.

<table>
<thead>
<tr>
<th>$a$</th>
<th>$\pi - \pi x = a$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\pi - \pi x = 0$</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>$\pi - \pi x = \frac{\pi}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$\pi - \pi x = \pi$</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{3\pi}{2}$</td>
<td>$\pi - \pi x = \frac{3\pi}{2}$</td>
<td>$-\frac{1}{2}$</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>$\pi - \pi x = 2\pi$</td>
<td>-1</td>
</tr>
</tbody>
</table>

Substituting these $x$-values into $g(x)$, we generate the graph below and find the period to be $1 - (-1) = 2$. The associated sine curve, $y = \frac{\sin(\pi - \pi x) - 5}{3}$, is dotted in as a reference.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
<th>$(x, g(x))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>undefined</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$-\frac{1}{3}$</td>
<td>($\frac{1}{2}$, $-\frac{1}{3}$)</td>
</tr>
<tr>
<td>0</td>
<td>undefined</td>
<td></td>
</tr>
<tr>
<td>$-\frac{1}{2}$</td>
<td>-2</td>
<td>($-\frac{1}{2}$, -2)</td>
</tr>
<tr>
<td>-1</td>
<td>undefined</td>
<td></td>
</tr>
</tbody>
</table>

One cycle of $y = \frac{\csc(\pi - \pi x) - 5}{3}$. □
Before moving on, we note that it is possible to speak of the period, phase shift and vertical shift of secant and cosecant graphs and use even/odd identities to put them in a form similar to the sinusoid forms mentioned in Theorem 10.23. Since these quantities match those of the corresponding cosine and sine curves, we do not spell this out explicitly. Finally, since the ranges of secant and cosecant are unbounded, there is no amplitude associated with these curves.

10.5.3 Graphs of the Tangent and Cotangent Functions

Finally, we turn our attention to the graphs of the tangent and cotangent functions. When constructing a table of values for the tangent function, we see that \( J(x) = \tan(x) \) is undefined at \( x = \frac{\pi}{2} \) and \( x = \frac{3\pi}{2} \), in accordance with our findings in Section 10.3.1. As \( x \to \frac{\pi}{2}^- \), \( \sin(x) \to 1^- \) and \( \cos(x) \to 0^+ \), so that \( \tan(x) = \frac{\sin(x)}{\cos(x)} \to \infty \) producing a vertical asymptote at \( x = \frac{\pi}{2} \). Using a similar analysis, we get that as \( x \to \frac{\pi}{2}^+ \), \( \tan(x) \to -\infty \); as \( x \to \frac{3\pi}{2}^- \), \( \tan(x) \to \infty \); and as \( x \to \frac{3\pi}{2}^+ \), \( \tan(x) \to -\infty \). Plotting this information and performing the usual 'copy and paste' produces:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \tan(x) )</th>
<th>( (x, \tan(x)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
<td>1</td>
<td>( \left( \frac{\pi}{4}, 1 \right) )</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>undefined</td>
<td></td>
</tr>
<tr>
<td>( \frac{3\pi}{4} )</td>
<td>-1</td>
<td>( \left( \frac{3\pi}{4}, -1 \right) )</td>
</tr>
<tr>
<td>( \pi )</td>
<td>0</td>
<td>( (\pi, 0) )</td>
</tr>
<tr>
<td>( \frac{5\pi}{4} )</td>
<td>1</td>
<td>( \left( \frac{5\pi}{4}, 1 \right) )</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>undefined</td>
<td></td>
</tr>
<tr>
<td>( \frac{7\pi}{4} )</td>
<td>-1</td>
<td>( \left( \frac{7\pi}{4}, -1 \right) )</td>
</tr>
<tr>
<td>2( \pi )</td>
<td>0</td>
<td>( (2\pi, 0) )</td>
</tr>
</tbody>
</table>

The graph of \( y = \tan(x) \) over \([0, 2\pi]\).
From the graph, it appears as if the tangent function is periodic with period $\pi$. To prove that this is the case, we appeal to the sum formula for tangents. We have:

$$\tan(x + \pi) = \frac{\tan(x) + \tan(\pi)}{1 - \tan(x)\tan(\pi)} = \frac{\tan(x) + 0}{1 - (\tan(x))(0)} = \tan(x),$$

which tells us the period of $\tan(x)$ is at most $\pi$. To show that it is exactly $\pi$, suppose $p$ is a positive real number so that $\tan(x + p) = \tan(x)$ for all real numbers $x$. For $x = 0$, we have $\tan(p) = \tan(0 + p) = \tan(0) = 0$, which means $p$ is a multiple of $\pi$. The smallest positive multiple of $\pi$ is $\pi$ itself, so we have established the result. We take as our fundamental cycle for $y = \tan(x)$ the interval $(-\pi/2, \pi/2)$, and use as our ‘quarter marks’ $x = -\pi/4, -\pi/2, 0, \pi/4$ and $\pi/2$. From the graph, we see confirmation of our domain and range work in Section 10.3.1.

It should be no surprise that $K(x) = \cot(x)$ behaves similarly to $J(x) = \tan(x)$. Plotting $\cot(x)$ over the interval $[0, 2\pi]$ results in the graph below.

From these data, it clearly appears as if the period of $\cot(x)$ is $\pi$, and we leave it to the reader to prove this.\(^\text{14}\) We take as one fundamental cycle the interval $(0, \pi)$ with quarter marks: $x = 0, \pi/4, \pi/2, 3\pi/4$ and $\pi$. A more complete graph of $y = \cot(x)$ is below, along with the fundamental cycle highlighted as usual. Once again, we see the domain and range of $K(x) = \cot(x)$ as read from the graph matches with what we found analytically in Section 10.3.1.

\(^\text{14}\)Certainly, mimicking the proof that the period of $\tan(x)$ is an option; for another approach, consider transforming $\tan(x)$ to $\cot(x)$ using identities.
The graph of $y = \cot(x)$.

The properties of the tangent and cotangent functions are summarized below. As with Theorem 10.24, each of the results below can be traced back to properties of the cosine and sine functions and the definition of the tangent and cotangent functions as quotients thereof.

**Theorem 10.25. Properties of the Tangent and Cotangent Functions**

- The function $J(x) = \tan(x)$
  - has domain $\{x : x \neq \frac{\pi}{2} + \pi k, \ k \text{ is an integer}\} = \bigcup_{k=-\infty}^{\infty} \left( \frac{(2k + 1)\pi}{2}, \frac{(2k + 3)\pi}{2} \right)$
  - has range $(-\infty, \infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period $\pi$

- The function $K(x) = \cot(x)$
  - has domain $\{x : x \neq \pi k, \ k \text{ is an integer}\} = \bigcup_{k=-\infty}^{\infty} (k\pi, (k + 1)\pi)$
  - has range $(-\infty, \infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period $\pi$
Example 10.5.5. Graph one cycle of the following functions. Find the period.

1. \( f(x) = 1 - \tan\left(\frac{x}{2}\right) \).

2. \( g(x) = 2 \cot\left(\frac{\pi}{2}x + \pi\right) + 1 \).

Solution.

1. We proceed as we have in all of the previous graphing examples by setting the argument of tangent in \( f(x) = 1 - \tan\left(\frac{x}{2}\right) \), namely \( \frac{x}{2} \), equal to each of the ‘quarter marks’ \( -\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \text{ and } \frac{\pi}{2} \), and solving for \( x \).

<table>
<thead>
<tr>
<th>( a )</th>
<th>( \frac{x}{2} = a )</th>
<th>( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -\frac{\pi}{2} )</td>
<td>( \frac{x}{2} = -\frac{\pi}{2} )</td>
<td>( -\pi )</td>
</tr>
<tr>
<td>( -\frac{\pi}{4} )</td>
<td>( \frac{x}{2} = -\frac{\pi}{4} )</td>
<td>( -\frac{\pi}{2} )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( \frac{x}{2} = 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
<td>( \frac{x}{2} = \frac{\pi}{4} )</td>
<td>( \frac{\pi}{2} )</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>( \frac{x}{2} = \frac{\pi}{2} )</td>
<td>( \pi )</td>
</tr>
</tbody>
</table>

Substituting these \( x \)-values into \( f(x) \), we find points on the graph and the vertical asymptotes.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( (x, f(x)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -\pi )</td>
<td>undefined</td>
<td>( (-\pi, 2) )</td>
</tr>
<tr>
<td>( -\pi/2 )</td>
<td>2</td>
<td>( (-\pi/2, 2) )</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>( (0, 1) )</td>
</tr>
<tr>
<td>( \pi/2 )</td>
<td>0</td>
<td>( (\pi/2, 0) )</td>
</tr>
<tr>
<td>( \pi )</td>
<td>undefined</td>
<td>( )</td>
</tr>
</tbody>
</table>

We see that the period is \( \pi - (-\pi) = 2\pi \).

2. The ‘quarter marks’ for the fundamental cycle of the cotangent curve are \( 0, \frac{\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}, \text{ and } \pi \). To graph \( g(x) = 2 \cot\left(\frac{\pi}{2}x + \pi\right) + 1 \), we begin by setting \( \frac{\pi}{2}x + \pi \) equal to each quarter mark and solving for \( x \).
We now use these $x$-values to generate our graph.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
<th>$(x, g(x))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td>undefined</td>
<td></td>
</tr>
<tr>
<td>$-\frac{3}{2}$</td>
<td>$3$</td>
<td>$\left(-\frac{3}{2}, 3\right)$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$1$</td>
<td>$(-1, 1)$</td>
</tr>
<tr>
<td>$-\frac{1}{2}$</td>
<td>$-1$</td>
<td>$\left(-\frac{1}{2}, -1\right)$</td>
</tr>
<tr>
<td>$0$</td>
<td>undefined</td>
<td></td>
</tr>
</tbody>
</table>

We find the period to be $0 - (-2) = 2$.

As with the secant and cosecant functions, it is possible to extend the notion of period, phase shift and vertical shift to the tangent and cotangent functions as we did for the cosine and sine functions in Theorem 10.23. Since the number of classical applications involving sinusoids far outnumber those involving tangent and cotangent functions, we omit this. The ambitious reader is invited to formulate such a theorem, however.
10.6 The Inverse Trigonometric Functions

As the title indicates, in this section we concern ourselves with finding inverses of the (circular) trigonometric functions. Our immediate problem is that, owing to their periodic nature, none of the six circular functions is one-to-one. To remedy this, we restrict the domains of the circular functions in the same way we restricted the domain of the quadratic function in Example 5.2.3 in Section 5.2 to obtain a one-to-one function. We first consider $f(x) = \cos(x)$. Choosing the interval $[0, \pi]$ allows us to keep the range as $[-1, 1]$ as well as the properties of being smooth and continuous.

Recalling from Section 5.2 that the inverse of a function $f$ is typically denoted $f^{-1}$. For this reason, some textbooks use the notation $f^{-1}(x) = \cos^{-1}(x)$ for the inverse of $f(x) = \cos(x)$. The obvious pitfall here is our convention of writing $(\cos(x))^2$ as $\cos^2(x)$, $(\cos(x))^3$ as $\cos^3(x)$ and so on. It is far too easy to confuse $\cos^{-1}(x)$ with $\frac{1}{\cos(x)} = \sec(x)$ so we will not use this notation in our text. Instead, we use the notation $f^{-1}(x) = \arccos(x)$, read ‘arc-cosine of $x$’. To understand the ‘arc’ in ‘arccosine’, recall that an inverse function, by definition, reverses the process of the original function. The function $f(t) = \cos(t)$ takes a real number input $t$, associates it with the angle $\theta = t$ radians, and returns the value $\cos(\theta)$. Digging deeper, we have that $\cos(\theta) = \cos(t)$ is the $x$-coordinate of the terminal point on the Unit Circle of an oriented arc of length $|t|$ whose initial point is $(1, 0)$. Hence, we may view the inputs to $f(t) = \cos(t)$ as oriented arcs and the outputs as $x$-coordinates on the Unit Circle. The function $f^{-1}$, then, would take $x$-coordinates on the Unit Circle and return oriented arcs, hence the ‘arc’ in arccosine. Below are the graphs of $f(x) = \cos(x)$ and $f^{-1}(x) = \arccos(x)$, where we obtain the latter from the former by reflecting it across the line $y = x$, in accordance with Theorem 5.3.

---

1But be aware that many books do! As always, be sure to check the context!

2See page 704 if you need a review of how we associate real numbers with angles in radian measure.
We restrict \( g(x) = \sin(x) \) in a similar manner, although the interval of choice is \([-\frac{\pi}{2}, \frac{\pi}{2}]\).

Restricting the domain of \( f(x) = \sin(x) \) to \([-\frac{\pi}{2}, \frac{\pi}{2}]\).

It should be no surprise that we call \( g^{-1}(x) = \arcsin(x) \), which is read ‘arc-sine of \( x \).

\[
g(x) = \sin(x), \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.
\]

\[
g^{-1}(x) = \arcsin(x).
\]

We list some important facts about the arccosine and arcsine functions in the following theorem.

**Theorem 10.26. Properties of the Arccosine and Arcsine Functions**

- **Properties of \( F(x) = \arccos(x) \)**
  - Domain: \([-1, 1]\)
  - Range: \([0, \pi]\)
  - \( \arccos(x) = t \) if and only if \( 0 \leq t \leq \pi \) and \( \cos(t) = x \)
  - \( \cos(\arccos(x)) = x \) provided \(-1 \leq x \leq 1\)
  - \( \arccos(\cos(x)) = x \) provided \( 0 \leq x \leq \pi \)

- **Properties of \( G(x) = \arcsin(x) \)**
  - Domain: \([-1, 1]\)
  - Range: \([-\frac{\pi}{2}, \frac{\pi}{2}]\]
  - \( \arcsin(x) = t \) if and only if \( -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \) and \( \sin(t) = x \)
  - \( \sin(\arcsin(x)) = x \) provided \(-1 \leq x \leq 1\)
  - \( \arcsin(\sin(x)) = x \) provided \(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\)
  - additionally, arcsine is odd
Everything in Theorem 10.26 is a direct consequence of the facts that \( f(x) = \cos(x) \) for \( 0 \leq x \leq \pi \) and \( F(x) = \arccos(x) \) are inverses of each other as are \( g(x) = \sin(x) \) for \( -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \) and \( G(x) = \arcsin(x) \). It’s about time for an example.

**Example 10.6.1.**

1. Find the exact values of the following.
   
   (a) \( \arccos \left( \frac{1}{2} \right) \)  
   (b) \( \arcsin \left( \frac{\sqrt{2}}{2} \right) \)  
   (c) \( \arccos \left( -\frac{\sqrt{2}}{2} \right) \)  
   (d) \( \arcsin \left( -\frac{1}{2} \right) \)  
   (e) \( \arccos \left( \cos \left( \frac{\pi}{6} \right) \right) \)  
   (f) \( \arccos \left( \cos \left( \frac{11\pi}{6} \right) \right) \)  
   (g) \( \cos \left( \arccos \left( -\frac{3}{5} \right) \right) \)  
   (h) \( \sin \left( \arccos \left( -\frac{3}{5} \right) \right) \)

2. Rewrite the following as algebraic expressions of \( x \) and state the domain on which the equivalence is valid.
   
   (a) \( \tan \left( \arccos \left( x \right) \right) \)  
   (b) \( \cos \left( 2 \arcsin \left( x \right) \right) \)

**Solution.**

1. (a) To find \( \arccos \left( \frac{1}{2} \right) \), we need to find the real number \( t \) (or, equivalently, an angle measuring \( t \) radians) which lies between 0 and \( \pi \) with \( \cos(t) = \frac{1}{2} \). We know \( t = \frac{\pi}{3} \) meets these criteria, so \( \arccos \left( \frac{1}{2} \right) = \frac{\pi}{3} \).

   (b) The value of \( \arcsin \left( \frac{\sqrt{2}}{2} \right) \) is a real number \( t \) between \( -\frac{\pi}{2} \) and \( \frac{\pi}{2} \) with \( \sin(t) = \frac{\sqrt{2}}{2} \). The number we seek is \( t = \frac{\pi}{4} \). Hence, \( \arcsin \left( \frac{\sqrt{2}}{2} \right) = \frac{\pi}{4} \).

   (c) The number \( t = \arccos \left( -\frac{\sqrt{2}}{2} \right) \) lies in the interval \( [0, \pi] \) with \( \cos(t) = -\frac{\sqrt{2}}{2} \). Our answer is \( \arccos \left( -\frac{\sqrt{2}}{2} \right) = \frac{3\pi}{4} \).

   (d) To find \( \arcsin \left( -\frac{1}{2} \right) \), we seek the number \( t \) in the interval \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \) with \( \sin(t) = -\frac{1}{2} \). The answer is \( t = -\frac{\pi}{6} \) so that \( \arcsin \left( -\frac{1}{2} \right) = -\frac{\pi}{6} \).

   (e) Since \( 0 \leq \frac{\pi}{6} \leq \pi \), we could simply invoke Theorem 10.26 to get \( \arccos \left( \cos \left( \frac{\pi}{6} \right) \right) = \frac{\pi}{6} \). However, in order to make sure we understand why this is the case, we choose to work the example through using the definition of arccosine. Working from the inside out, \( \arccos \left( \cos \left( \frac{\pi}{6} \right) \right) = \arccos \left( \frac{\sqrt{3}}{2} \right) \). Now, \( \arccos \left( \frac{\sqrt{3}}{2} \right) \) is the real number \( t \) with \( 0 \leq t \leq \pi \) and \( \cos(t) = \frac{\sqrt{3}}{2} \). We find \( t = \frac{\pi}{6} \), so that \( \arccos \left( \cos \left( \frac{\pi}{6} \right) \right) = \frac{\pi}{6} \).
(f) Since $\frac{11\pi}{6}$ does not fall between 0 and $\pi$, Theorem 10.26 does not apply. We are forced to work through from the inside out starting with $\arccos \left( \cos \left( \frac{11\pi}{6} \right) \right) = \arccos \left( \frac{-\sqrt{3}}{2} \right)$. From the previous problem, we know $\arccos \left( \frac{-\sqrt{3}}{2} \right) = \frac{\pi}{6}$. Hence, $\arccos \left( \cos \left( \frac{11\pi}{6} \right) \right) = \frac{\pi}{6}.

(g) One way to simplify $\cos \left( \arccos \left( -\frac{3}{5} \right) \right)$ is to use Theorem 10.26 directly. Since $-\frac{3}{5}$ is between $-1$ and $1$, we have that $\cos \left( \arccos \left( -\frac{3}{5} \right) \right) = -\frac{3}{5}$ and we are done. However, as before, to really understand why this cancellation occurs, we let $t = \arccos \left( -\frac{3}{5} \right)$. Then, by definition, $\cos(t) = -\frac{3}{5}$. Hence, $\cos \left( \arccos \left( -\frac{3}{5} \right) \right) = \cos(t) = -\frac{3}{5}$, and we are finished in (nearly) the same amount of time.

(h) As in the previous example, we let $t = \arccos \left( -\frac{3}{5} \right)$ so that $\cos(t) = -\frac{3}{5}$ for some $t$ where $0 \leq t \leq \pi$. Since $\cos(t) < 0$, we can narrow this down a bit and conclude that $\frac{\pi}{2} < t < \pi$, so that $t$ corresponds to an angle in Quadrant II. In terms of $t$, then, we need to find $\sin \left( \arccos \left( -\frac{3}{5} \right) \right) = \sin(t)$. Using the Pythagorean Identity $\cos^2(t) + \sin^2(t) = 1$, we get $\left( -\frac{3}{5} \right)^2 + \sin^2(t) = 1$ or $\sin(t) = \pm \frac{4}{5}$. Since $t$ corresponds to a Quadrants II angle, we choose $\sin(t) = \frac{4}{5}$. Hence, $\sin \left( \arccos \left( -\frac{3}{5} \right) \right) = \frac{4}{5}$.

2. (a) We begin this problem in the same manner we began the previous two problems. To help us see the forest for the trees, we let $t = \arccos(x)$, so our goal is to find a way to express $\tan \left( \arccos \left( x \right) \right) = \tan(t)$ in terms of $x$. Since $t = \arccos(x)$, we know $\cos(t) = x$ where $0 \leq t \leq \pi$, but since we are after an expression for $\tan(t)$, we know we need to throw out $t = \frac{\pi}{2}$ from consideration. Hence, either $0 \leq t < \frac{\pi}{2}$ or $\frac{\pi}{2} < t \leq \pi$ so that, geometrically, $t$ corresponds to an angle in Quadrant I or Quadrant II. One approach to finding $\tan(t)$ is to use the quotient identity $\tan(t) = \frac{\sin(t)}{\cos(t)}$. Substituting $\cos(t) = x$ into the Pythagorean Identity $\cos^2(t) + \sin^2(t) = 1$ gives $x^2 + \sin^2(t) = 1$, from which we get $\sin(t) = \pm \sqrt{1 - x^2}$. Since $t$ corresponds to angles in Quadrants I and II, $\sin(t) \geq 0$, so we choose $\sin(t) = \sqrt{1 - x^2}$. Thus,

$$\tan(t) = \frac{\sin(t)}{\cos(t)} = \frac{\sqrt{1 - x^2}}{x}$$

To determine the values of $x$ for which this equivalence is valid, we consider our substitution $t = \arccos(x)$. Since the domain of $\arccos(x)$ is $[-1, 1]$, we know we must restrict $-1 \leq x \leq 1$. Additionally, since we had to discard $t = \frac{\pi}{2}$, we need to discard $x = \cos \left( \frac{\pi}{2} \right) = 0$. Hence, $\tan \left( \arccos \left( x \right) \right) = \frac{\sqrt{1 - x^2}}{x}$ is valid for $x$ in $[-1, 0) \cup (0, 1]$.

(b) We proceed as in the previous problem by writing $t = \arcsin(x)$ so that $t$ lies in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\sin(t) = x$. We aim to express $\cos \left( 2 \arcsin \left( x \right) \right) = \cos(2t)$ in terms of $x$. Since $\cos(2t)$ is defined everywhere, we get no additional restrictions on $t$ as we did in the previous problem. We have three choices for rewriting $\cos(2t)$: $\cos^2(t) - \sin^2(t)$, $2 \cos^2(t) - 1$ and $1 - 2 \sin^2(t)$. Since we know $x = \sin(t)$, it is easiest to use the last form:

$$\cos \left( 2 \arcsin \left( x \right) \right) = \cos(2t) = 1 - 2 \sin^2(t) = 1 - 2x^2$$

\(^3\)Alternatively, we could use the identity: $1 + \tan^2(t) = \sec^2(t)$. Since $x = \cos(t)$, $\sec(t) = \frac{1}{\cos(t)} = \frac{1}{x}$. The reader is invited to work through this approach to see what, if any, difficulties arise.
10.6 The Inverse Trigonometric Functions

To find the restrictions on \( x \), we once again appeal to our substitution \( t = \arcsin(x) \). Since \( \arcsin(x) \) is defined only for \(-1 \leq x \leq 1\), the equivalence \( \cos(2 \arcsin(x)) = 1 - 2x^2 \) is valid only on \([-1, 1]\).

A few remarks about Example 10.6.1 are in order. Most of the common errors encountered in dealing with the inverse circular functions come from the need to restrict the domains of the original functions so that they are one-to-one. One instance of this phenomenon is the fact that \( \arccos(\cos(\frac{11\pi}{6})) = \frac{\pi}{6} \) as opposed to \( \frac{11\pi}{6} \). This is the exact same phenomenon discussed in Section 5.2 when we saw \( \sqrt{(-2)^2} = 2 \) as opposed to \(-2\). Additionally, even though the expression we arrived at in part 2b above, namely \( 1 - 2x^2 \), is defined for all real numbers, the equivalence \( \cos(2 \arcsin(x)) = 1 - 2x^2 \) is valid for only \(-1 \leq x \leq 1\). This is akin to the fact that while the expression \( x \) is defined for all real numbers, the equivalence \( (\sqrt{x})^2 = x \) is valid only for \( x \geq 0 \). For this reason, it pays to be careful when we determine the intervals where such equivalences are valid.

The next pair of functions we wish to discuss are the inverses of tangent and cotangent, which are named arctangent and arccotangent, respectively. First, we restrict \( f(x) = \tan(x) \) to its fundamental cycle on \((-\frac{\pi}{2}, \frac{\pi}{2})\) to obtain \( f^{-1}(x) = \arctan(x) \). Among other things, note that the vertical asymptotes \( x = -\frac{\pi}{2} \) and \( x = \frac{\pi}{2} \) of the graph of \( f(x) = \tan(x) \) become the horizontal asymptotes \( y = -\frac{\pi}{2} \) and \( y = \frac{\pi}{2} \) of the graph of \( f^{-1}(x) = \arctan(x) \).

Next, we restrict \( g(x) = \cot(x) \) to its fundamental cycle on \((0, \pi)\) to obtain \( g^{-1}(x) = \arccot(x) \). Once again, the vertical asymptotes \( x = 0 \) and \( x = \pi \) of the graph of \( g(x) = \cot(x) \) become the horizontal asymptotes \( y = 0 \) and \( y = \pi \) of the graph of \( g^{-1}(x) = \arccot(x) \). We show these graphs on the next page and list some of the basic properties of the arctangent and arccotangent functions.
Theorem 10.27. Properties of the Arctangent and Arccotangent Functions

- Properties of \( F(x) = \arctan(x) \)
  - Domain: \((-\infty, \infty)\)
  - Range: \((-\frac{\pi}{2}, \frac{\pi}{2})\)
  - as \( x \to -\infty \), \( \arctan(x) \to -\frac{\pi}{2}^+ \); as \( x \to \infty \), \( \arctan(x) \to \frac{\pi}{2}^- \)
  - \( \arctan(x) = t \) if and only if \(-\frac{\pi}{2} < t < \frac{\pi}{2}\) and \( \tan(t) = x \)
  - \( \arctan(x) = \arccot\left(\frac{1}{x}\right) \) for \( x > 0 \)
  - \( \tan\left(\arctan(x)\right) = x \) for all real numbers \( x \)
  - \( \arctan(\tan(x)) = x \) provided \(-\frac{\pi}{2} < x < \frac{\pi}{2}\)
  - additionally, \( \arctan \) is odd

- Properties of \( G(x) = \arccot(x) \)
  - Domain: \((-\infty, \infty)\)
  - Range: \((0, \pi)\)
  - as \( x \to -\infty \), \( \arccot(x) \to \pi^- \); as \( x \to \infty \), \( \arccot(x) \to 0^+ \)
  - \( \arccot(x) = t \) if and only if \( 0 < t < \pi \) and \( \cot(t) = x \)
  - \( \arccot(x) = \arctan\left(\frac{1}{x}\right) \) for \( x > 0 \)
  - \( \cot\left(\arccot(x)\right) = x \) for all real numbers \( x \)
  - \( \arccot(\cot(x)) = x \) provided \( 0 < x < \pi \)
Example 10.6.2.

1. Find the exact values of the following.
   
   (a) \( \arctan(\sqrt{3}) \)  
   
   (b) \( \arccot(-\sqrt{3}) \)  
   
   (c) \( \cot(\arccot(-5)) \)  
   
   (d) \( \sin(\arctan(-\frac{3}{4})) \)  

2. Rewrite the following as algebraic expressions of \( x \) and state the domain on which the equivalence is valid.
   
   (a) \( \tan(2 \arctan(x)) \)  
   
   (b) \( \cos(\arccot(2x)) \)  

Solution.

1. (a) We know \( \arctan(\sqrt{3}) \) is the real number \( t \) between \( -\frac{\pi}{2} \) and \( \frac{\pi}{2} \) with \( \tan(t) = \sqrt{3} \). We find \( t = \frac{\pi}{3} \), so \( \arctan(\sqrt{3}) = \frac{\pi}{3} \).

(b) The real number \( t = \arccot(-\sqrt{3}) \) lies in the interval \((0, \pi)\) with \( \cot(t) = -\sqrt{3} \). We get \( \arccot(-\sqrt{3}) = \frac{5\pi}{6} \).

(c) We can apply Theorem 10.27 directly and obtain \( \cot(\arccot(-5)) = -5 \). However, working it through provides us with yet another opportunity to understand why this is the case. Letting \( t = \arccot(-5) \), we have that \( t \) belongs to the interval \((0, \pi)\) and \( \cot(t) = -5 \). Hence, \( \cot(\arccot(-5)) = \cot(t) = -5 \).

(d) We start simplifying \( \sin(\arctan(-\frac{3}{4})) \) by letting \( t = \arctan(-\frac{3}{4}) \). Then \( \tan(t) = -\frac{3}{4} \) for some \( -\frac{\pi}{2} < t < \frac{\pi}{2} \). Since \( \tan(t) < 0 \), we know, in fact, \( -\frac{\pi}{2} < t < 0 \). One way to proceed is to use The Pythagorean Identity, \( 1 + \cot^2(t) = \csc^2(t) \), since this relates the reciprocals of \( \tan(t) \) and \( \sin(t) \) and is valid for all \( t \) under consideration.\(^4\) From \( \tan(t) = -\frac{3}{4} \), we get \( \cot(t) = -\frac{4}{3} \). Substituting, we get \( 1 + \left(-\frac{4}{3}\right)^2 = \csc^2(t) \) so that \( \csc(t) = \frac{\pm 5}{3} \). Since \( -\frac{\pi}{2} < t < 0 \), we choose \( \csc(t) = -\frac{5}{3} \), so \( \sin(t) = -\frac{3}{5} \). Hence, \( \sin(\arctan(-\frac{3}{4})) = -\frac{3}{5} \).

2. (a) If we let \( t = \arctan(x) \), then \( -\frac{\pi}{2} < t < \frac{\pi}{2} \) and \( \tan(t) = x \). We look for a way to express \( \tan(2 \arctan(x)) = \tan(2t) \) in terms of \( x \). Before we get started using identities, we note that \( \tan(2t) \) is undefined when \( 2t = \frac{\pi}{2} + \pi k \) for integers \( k \). Dividing both sides of this equation by 2 tells us we need to exclude values of \( t \) where \( t = \frac{\pi}{4} + \frac{\pi}{2} k \), where \( k \) is an integer. The only members of this family which lie in \( (-\frac{\pi}{2}, \frac{\pi}{2}) \) are \( t = \pm \frac{\pi}{4} \), which means the values of \( t \) under consideration are \( (-\frac{\pi}{4}, -\frac{\pi}{4}) \cup (-\frac{\pi}{4}, \frac{\pi}{4}) \cup (\frac{\pi}{4}, \frac{3\pi}{4}) \). Returning to \( \arctan(2t) \), we note the double angle identity \( \tan(2t) = \frac{2 \tan(t)}{1 - \tan^2(t)} \), is valid for all the values of \( t \) under consideration, hence we get

\[
\tan(2 \arctan(x)) = \tan(2t) = \frac{2 \tan(t)}{1 - \tan^2(t)} = \frac{2x}{1 - x^2}
\]

\(^4\)It’s always a good idea to make sure the identities used in these situations are valid for all values \( t \) under consideration. Check our work back in Example 10.6.1. Were the identities we used there valid for all \( t \) under consideration? A pedantic point, to be sure, but what else do you expect from this book?
To find where this equivalence is valid we check back with our substitution $t = \arctan(x)$. Since the domain of $\arctan(x)$ is all real numbers, the only exclusions come from the values of $t$ we discarded earlier, $t = \pm \frac{\pi}{4}$. Since $x = \tan(t)$, this means we exclude $x = \tan \left( \pm \frac{\pi}{4} \right) = \pm 1$. Hence, the equivalence $\tan(2\arctan(x)) = \frac{2x}{1-x^2}$ holds for all $x$ in $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

(b) To get started, we let $t = \arccot(2x)$ so that $\cot(t) = 2x$ where $0 < t < \pi$. In terms of $t$, $\cos(\arccot(2x)) = \cos(t)$, and our goal is to express the latter in terms of $x$. Since $\cos(t)$ is always defined, there are no additional restrictions on $t$, so we can begin using identities to relate $\cot(t)$ to $\cos(t)$. The identity $\cot(t) = \frac{\cos(t)}{\sin(t)}$ is valid for $t$ in $(0, \pi)$, so our strategy is to obtain $\sin(t)$ in terms of $x$, then write $\cos(t) = \cot(t)\sin(t)$. The identity $1 + \cot^2(t) = \csc^2(t)$ holds for all $t$ in $(0, \pi)$ and relates $\cot(t)$ and $\csc(t) = \frac{1}{\sin(t)}$. Substituting $\cot(t) = 2x$, we get $1 + (2x)^2 = \csc^2(t)$, or $\csc(t) = \pm \sqrt{4x^2 + 1}$. Since $t$ is between 0 and $\pi$, $\csc(t) > 0$, so $\csc(t) = \sqrt{4x^2 + 1}$ which gives $\sin(t) = \frac{1}{\sqrt{4x^2 + 1}}$. Hence,

$$\cos(\arccot(2x)) = \cos(t) = \cot(t)\sin(t) = \frac{2x}{\sqrt{4x^2 + 1}}$$

Since $\arccot(2x)$ is defined for all real numbers $x$ and we encountered no additional restrictions on $t$, we have $\cos(\arccot(2x)) = \frac{2x}{\sqrt{4x^2 + 1}}$ for all real numbers $x$.

The last two functions to invert are secant and cosecant. A portion of each of their graphs, which were first discussed in Subsection 10.5.2, are given below with the fundamental cycles highlighted.

It is clear from the graph of secant that we cannot find one single continuous piece of its graph which covers its entire range of $(-\infty, -1] \cup [1, \infty)$ and restricts the domain of the function so that it is one-to-one. The same is true for cosecant. Thus in order to define the arcsecant and arccosecant functions, we must settle for a piecewise approach wherein we choose one piece to cover the top of the range, namely $[1, \infty)$, and another piece to cover the bottom, namely $(-\infty, -1]$. There are two generally accepted ways make these choices which restrict the domains of these functions so that they are one-to-one. One approach simplifies the Trigonometry associated with the inverse functions, but complicates the Calculus; the other makes the Calculus easier, but the Trigonometry less so. We present both points of view.
10.6 The Inverse Trigonometric Functions

10.6.1 Inverses of Secant and Cosecant: Trigonometry Friendly Approach

In this subsection, we restrict the secant and cosecant functions to coincide with the restrictions on cosine and sine, respectively. For \( f(x) = \sec(x) \), we restrict the domain to \( [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi] \)

\[
\begin{align*}
  f(x) &= \sec(x) \text{ on } [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi] \\
  f^{-1}(x) &= \text{arcsec}(x)
\end{align*}
\]

and we restrict \( g(x) = \csc(x) \) to \( [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}] \).

\[
\begin{align*}
  g(x) &= \csc(x) \text{ on } [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}] \\
  g^{-1}(x) &= \text{arccsc}(x)
\end{align*}
\]

Note that for both arcsecant and arccosecant, the domain is \( (-\infty, -1] \cup [1, \infty) \). Taking a page from Section 2.2, we can rewrite this as \( \{x : |x| \geq 1\} \). This is often done in Calculus textbooks, so we include it here for completeness. Using these definitions, we get the following properties of the arcsecant and arccosecant functions.
Theorem 10.28. Properties of the Arcsecant and Arccosecant Functions\(^a\)

- **Properties of** \(F(x) = \text{arcsec}(x)\)
  - **Domain:** \(\{x : |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)\)
  - **Range:** \([0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]\)
  - as \(x \to -\infty\), \(\text{arcsec}(x) \to \frac{\pi}{2}^+\); as \(x \to \infty\), \(\text{arcsec}(x) \to \frac{\pi}{2}^-\)
  - \(\text{arcsec}(x) = t\) if and only if \(0 \leq t < \frac{\pi}{2}\) or \(\frac{\pi}{2} < t \leq \pi\) and \(\sec(t) = x\)
  - \(\text{arcsec}(x) = \arccos\left(\frac{1}{x}\right)\) provided \(|x| \geq 1\)
  - \(\sec(\text{arcsec}(x)) = x\) provided \(|x| \geq 1\)
  - \(\text{arcsec}(\sec(x)) = x\) provided \(0 \leq x < \frac{\pi}{2}\) or \(\frac{\pi}{2} < x \leq \pi\)

- **Properties of** \(G(x) = \text{arccsc}(x)\)
  - **Domain:** \(\{x : |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)\)
  - **Range:** \((-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]\)
  - as \(x \to -\infty\), \(\text{arccsc}(x) \to 0^-\); as \(x \to \infty\), \(\text{arccsc}(x) \to 0^+\)
  - \(\text{arccsc}(x) = t\) if and only if \(-\frac{\pi}{2} \leq t < 0\) or \(0 < t \leq \frac{\pi}{2}\) and \(\csc(t) = x\)
  - \(\text{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)\) provided \(|x| \geq 1\)
  - \(\csc(\text{arccsc}(x)) = x\) provided \(|x| \geq 1\)
  - \(\text{arccsc}(\csc(x)) = x\) provided \(-\frac{\pi}{2} \leq x < 0\) or \(0 < x \leq \frac{\pi}{2}\)
  - additionally, arccosecant is odd

\(^a\)...assuming the “Trigonometry Friendly” ranges are used.

**Example 10.6.3.**

1. Find the exact values of the following.
   
   \[
   \begin{align*}
   (a) \ & \text{arcsec}(2) \quad (b) \ & \text{arccsc}(-2) \quad (c) \ & \text{arcsec} \left( \sec\left(\frac{5\pi}{4}\right) \right) \quad (d) \ & \cot \left( \text{arccsc} \left( -3 \right) \right)
   \end{align*}
   \]

2. Rewrite the following as algebraic expressions of \(x\) and state the domain on which the equivalence is valid.
   
   \[
   \begin{align*}
   (a) \ & \tan(\text{arcsec}(x)) \quad (b) \ & \cos(\text{arccsc}(4x))
   \end{align*}
   \]
### Solution.

1. (a) Using Theorem 10.28, we have \( \arccsc(2) = \arccos \left( \frac{1}{2} \right) = \frac{\pi}{3} \).

(b) Once again, Theorem 10.28 comes to our aid giving \( \arccsc(-2) = \arcsin \left( -\frac{1}{2} \right) = -\frac{\pi}{6} \).

(c) Since \( \frac{5\pi}{4} \) doesn’t fall between 0 and \( \frac{3\pi}{2} \) or \( \frac{3\pi}{2} \) and \( \pi \), we cannot use the inverse property stated in Theorem 10.28. We can, nevertheless, begin by working ‘inside out’ which yields \( \arccsc \left( \sec \left( \frac{5\pi}{4} \right) \right) = \arccsc(-\sqrt{2}) = \arccos \left( -\frac{\sqrt{2}}{2} \right) = \frac{3\pi}{4} \).

(d) One way to begin to simplify \( \cot (\arccsc (-3)) \) is to let \( t = \arccsc(-3) \). Then, \( \csc(t) = -3 \) and, since this is negative, we have that \( t \) lies in the interval \( [-\pi, 0] \). We are after \( \cot (\arccsc(-3)) = \cot(t) \), so we use the Pythagorean Identity \( 1 + \cot^2(t) = \csc^2(t) \). Substituting, we have \( 1 + \cot^2(t) = (-3)^2 \), or \( \cot(t) = \pm \sqrt{8} = \pm 2\sqrt{2} \). Since \( -\frac{\pi}{2} \leq t < 0 \), \( \cot(t) < 0 \), so we get \( \cot (\arccsc(-3)) = -2\sqrt{2} \).

2. (a) We begin simplifying \( \tan(\arccsc(x)) \) by letting \( t = \arccsc(x) \). Then, \( \sec(t) = x \) for \( t \) in \( [0, \frac{\pi}{2}] \cup (\frac{\pi}{2}, \pi] \), and we seek a formula for \( \tan(t) \). Since \( \tan(t) \) is defined for all \( t \) values under consideration, we have no additional restrictions on \( t \). To relate \( \sec(t) \) to \( \tan(t) \), we use the identity \( 1 + \tan^2(t) = \sec^2(t) \). This is valid for all values of \( t \) under consideration, and when we substitute \( \sec(t) = x \), we get \( 1 + \tan^2(t) = x^2 \). Hence, \( \tan(t) = \pm \sqrt{x^2 - 1} \).

If \( t \) belongs to \( [0, \frac{\pi}{2}] \) then \( \tan(t) \geq 0 \); if, on the the other hand, \( t \) belongs to \( (\frac{\pi}{2}, \pi] \) then \( \tan(t) \leq 0 \). As a result, we get a piecewise defined function for \( \tan(t) \)

\[
\tan(t) = \begin{cases} 
\sqrt{x^2 - 1}, & \text{if } 0 \leq t < \frac{\pi}{2} \\
-\sqrt{x^2 - 1}, & \text{if } \frac{\pi}{2} < t \leq \pi
\end{cases}
\]

Now we need to determine what these conditions on \( t \) mean for \( x \). Since \( x = \sec(t) \), when \( 0 \leq t < \frac{\pi}{2}, x \geq 1 \), and when \( \frac{\pi}{2} < t \leq \pi, x \leq -1 \). Since we encountered no further restrictions on \( t \), the equivalence below holds for all \( x \) in \( (-\infty, -1] \cup [1, \infty) \).

\[
\tan(\arccsc(x)) = \begin{cases} 
\sqrt{x^2 - 1}, & \text{if } x \geq 1 \\
-\sqrt{x^2 - 1}, & \text{if } x \leq -1
\end{cases}
\]

(b) To simplify \( \cos(\arccsc(4x)) \), we start by letting \( t = \arccsc(4x) \). Then \( \csc(t) = 4x \) for \( t \) in \( \left[ -\frac{\pi}{2}, 0 \right) \cup (0, \frac{\pi}{2}] \), and we now set about finding an expression for \( \cos(\arccsc(4x)) \) = \( \cos(t) \). Since \( \cos(t) \) is defined for all \( t \), we do not encounter any additional restrictions on \( t \). From \( \csc(t) = 4x \), we get \( \sin(t) = \frac{1}{4|x|} \), so to find \( \cos(t) \), we can make use if the identity \( \cos^2(t) + \sin^2(t) = 1 \). Substituting \( \sin(t) = \frac{1}{4|x|} \) gives \( \cos^2(t) + \left( \frac{1}{4|x|} \right)^2 = 1 \). Solving, we get

\[
\cos(t) = \pm \sqrt{\frac{16x^2 - 1}{16x^2}} = \pm \frac{\sqrt{16x^2 - 1}}{4|x|}
\]

Since \( t \) belongs to \( \left[ -\frac{\pi}{2}, 0 \right) \cup (0, \frac{\pi}{2}] \), we know \( \cos(t) \geq 0 \), so we choose \( \cos(t) = \frac{\sqrt{16x^2 - 1}}{4|x|} \).

(The absolute values here are necessary, since \( x \) could be negative.) To find the values for
which this equivalence is valid, we look back at our original substitution, \( t = \arccsc(4x) \). Since the domain of \( \arccsc(x) \) requires its argument \( x \) to satisfy \(|x| \geq 1\), the domain of \( \arccsc(4x) \) requires \(|4x| \geq 1\). Using Theorem 2.4, we rewrite this inequality and solve to get \( x \leq -\frac{1}{4} \) or \( x \geq \frac{1}{4} \). Since we had no additional restrictions on \( t \), the equivalence \( \cos(\arccsc(4x)) = \sqrt{\frac{16x^2-1}{4|x|}} \) holds for all \( x \) in \( (-\infty, -\frac{1}{4}] \cup [\frac{1}{4}, \infty) \).

10.6.2 Inverses of Secant and Cosecant: Calculus Friendly Approach

In this subsection, we restrict \( f(x) = \sec(x) \) to \( [0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2}) \)

and we restrict \( g(x) = \csc(x) \) to \((0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}] \).

Using these definitions, we get the following result.
Theorem 10.29. Properties of the Arcsecant and Arccosecant Functions

- Properties of $F(x) = \text{arcsec}(x)$
  - Domain: $\{ x : |x| \geq 1 \} = (-\infty, -1] \cup [1, \infty)$
  - Range: $[0, \frac{\pi}{2}) \cup \left[ \pi, \frac{3\pi}{2} \right)$
  - as $x \to -\infty$, $\text{arcsec}(x) \to \frac{3\pi}{2}$; as $x \to \infty$, $\text{arcsec}(x) \to \frac{\pi}{2}$
  - $\text{arcsec}(x) = t$ if and only if $0 \leq t < \frac{\pi}{2}$ or $\pi \leq t < \frac{3\pi}{2}$ and $\sec(t) = x$
  - $\text{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ for $x \geq 1$ only\(^{b}\)
  - $\sec(\text{arcsec}(x)) = x$ provided $|x| \geq 1$
  - $\text{arcsec}(\sec(x)) = x$ provided $0 \leq x < \frac{\pi}{2}$ or $\pi \leq x < \frac{3\pi}{2}$

- Properties of $G(x) = \text{arccsc}(x)$
  - Domain: $\{ x : |x| \geq 1 \} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left( 0, \frac{\pi}{2} \right] \cup \left( \pi, \frac{3\pi}{2} \right]$
  - as $x \to -\infty$, $\text{arccsc}(x) \to \frac{\pi}{2}$; as $x \to \infty$, $\text{arccsc}(x) \to 0^{+}$
  - $\text{arccsc}(x) = t$ if and only if $0 < t \leq \frac{\pi}{2}$ or $\pi < t \leq \frac{3\pi}{2}$ and $\csc(t) = x$
  - $\text{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$ for $x \geq 1$ only\(^{c}\)
  - $\csc(\text{arccsc}(x)) = x$ provided $|x| \geq 1$
  - $\text{arccsc}(\csc(x)) = x$ provided $0 < x \leq \frac{\pi}{2}$ or $\pi < x \leq \frac{3\pi}{2}$

\(^{a}\)...assuming the “Calculus Friendly” ranges are used.
\(^{b}\)Compare this with the similar result in Theorem 10.28.
\(^{c}\)Compare this with the similar result in Theorem 10.28.

Our next example is a duplicate of Example 10.6.3. The interested reader is invited to compare and contrast the solution to each.

Example 10.6.4.

1. Find the exact values of the following.

   (a) $\text{arcsec}(2)$  (b) $\text{arccsc}(-2)$  (c) $\text{arccsc}\left(\sec\left(\frac{5\pi}{4}\right)\right)$  (d) $\cot(\text{arccsc}(-3))$

2. Rewrite the following as algebraic expressions of $x$ and state the domain on which the equivalence is valid.

   (a) $\tan(\text{arcsec}(x))$  (b) $\cos(\text{arccsc}(4x))$
1. (a) Since $2 \geq 1$, we may invoke Theorem 10.29 to get $\arccsc(2) = \arccos \left( \frac{1}{2} \right) = \frac{\pi}{3}$.

(b) Unfortunately, $-2$ is not greater to or equal to 1, so we cannot apply Theorem 10.29 to $\arccsc(-2)$ and convert this into an arcsine problem. Instead, we appeal to the definition. The real number $t = \arccsc(-2)$ lies in $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ and satisfies $\csc(t) = -2$. The $t$ we’re after is $t = \frac{3\pi}{6}$, so $\arccsc(-2) = \frac{3\pi}{6}$.

(c) Since $\arccsc \left( \frac{5\pi}{4} \right)$ lies between $\pi$ and $\frac{3\pi}{2}$, we may apply Theorem 10.29 directly to simplify $\arccsc \left( \frac{5\pi}{4} \right)$. We can use the identity $1 + \cot^2(t) = \csc^2(t)$, we find $1 + \cot^2(t) = (-3)^2$ so that $\cot(t) = \pm \sqrt{8} = \pm 2\sqrt{2}$. Since $t$ is in the interval $\left(\pi, \frac{3\pi}{2}\right]$, we know $\cot(t) > 0$. Our answer is $\cot \left( \arccsc \left( -3 \right) \right) = 2\sqrt{2}$.

2. (a) We begin simplifying $\tan(\arccsc(x))$ by letting $t = \arccsc(x)$. Then, $\csc(t) = x$ for $t$ in $\left[0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$, and we seek a formula for $\tan(t)$. Since $\tan(t)$ is defined for all $t$ values under consideration, we have no additional restrictions on $t$. To relate $\csc(t)$ to $\tan(t)$, we use the identity $1 + \cot^2(t) = \csc^2(t)$. This is valid for all values of $t$ under consideration, and when we substitute $\csc(t) = x$, we get $1 + \tan^2(t) = x^2$. Hence, $\tan(t) = \pm \sqrt{x^2 - 1}$. Since $t$ lies in $\left[0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$, $\tan(t) \geq 0$, so we choose $\tan(t) = \sqrt{x^2 - 1}$. Since we found no additional restrictions on $t$, the equivalence $\tan(\arccsc(x)) = \sqrt{x^2 - 1}$ holds for all $x$ in the domain of $t = \arccsc(x)$, namely $(-\infty, -1] \cup [1, \infty)$.

(b) To simplify $\cos(\arccsc(4x))$, we start by letting $t = \arccsc(4x)$. Then $\csc(t) = 4x$ for $t$ in $\left[0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$, and we now set about finding an expression for $\cos(\arccsc(4x)) = \cos(t)$. Since $\cos(t)$ is defined for all $t$, we do not encounter any additional restrictions on $t$. From $\csc(t) = 4x$, we get $\sin(t) = \frac{1}{4x}$, so to find $\cos(t)$, we can make use if the identity $\cos^2(t) + \sin^2(t) = 1$. Substituting $\sin(t) = \frac{1}{4x}$ gives $\cos^2(t) + \left( \frac{1}{4x} \right)^2 = 1$. Solving, we get

$$\cos(t) = \pm \sqrt{\frac{16x^2 - 1}{16x^2}} = \frac{\pm \sqrt{16x^2 - 1}}{4|x|}$$

If $t$ lies in $\left(0, \frac{\pi}{2}\right)$, then $\cos(t) \geq 0$, and we choose $\cos(t) = \frac{\sqrt{16x^2 - 1}}{4|x|}$. Otherwise, $t$ belongs to $\left(\pi, \frac{3\pi}{2}\right]$ in which case $\cos(t) \leq 0$, so, we choose $\cos(t) = -\frac{\sqrt{16x^2 - 1}}{4|x|}$. This leads us to a (momentarily) piecewise defined function for $\cos(t)$

$$\cos(t) = \begin{cases} \frac{\sqrt{16x^2 - 1}}{4|x|}, & \text{if } 0 \leq t \leq \frac{\pi}{2} \\ -\frac{\sqrt{16x^2 - 1}}{4|x|}, & \text{if } \pi < t \leq \frac{3\pi}{2} \end{cases}$$
We now see what these restrictions mean in terms of $x$. Since $4x = \csc(t)$, we get that for $0 \leq t \leq \frac{\pi}{2}$, $4x \geq 1$, or $x \geq \frac{1}{4}$. In this case, we can simplify $|x| = x$ so

$$\cos(t) = \frac{\sqrt{16x^2 - 1}}{4|x|} = \frac{\sqrt{16x^2 - 1}}{4x}$$

Similarly, for $\pi < t \leq \frac{3\pi}{2}$, we get $4x \leq -1$, or $x \leq -\frac{1}{4}$. In this case, $|x| = -x$, so we also get

$$\cos(t) = -\frac{\sqrt{16x^2 - 1}}{4|x|} = -\frac{\sqrt{16x^2 - 1}}{4(-x)} = \frac{\sqrt{16x^2 - 1}}{4x}$$

Hence, in all cases, $\cos(\arccsc(4x)) = \frac{\sqrt{16x^2 - 1}}{4x}$, and this equivalence is valid for all $x$ in the domain of $t = \arccsc(4x)$, namely $(-\infty, -\frac{1}{4}] \cup \left[ \frac{1}{4}, \infty \right)$.

### 10.6.3 Calculators and the Inverse Circular Functions.

In the sections to come, we will have need to approximate the values of the inverse circular functions. On most calculators, only the arcsine, arccosine and arctangent functions are available and they are usually labeled as $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$, respectively. If we are asked to approximate these values, it is a simple matter to punch up the appropriate decimal on the calculator. If we are asked for an arccotangent, arcsecant or arccosecant, however, we often need to employ some ingenuity, as our next example illustrates.

**Example 10.6.5.**

1. Use a calculator to approximate the following values to four decimal places.

   (a) $\arccot(2)$  
   (b) $\text{arcsec}(5)$  
   (c) $\arccot(-2)$  
   (d) $\arccsc\left(-\frac{3}{2}\right)$

2. Find the domain and range of the following functions. Check your answers using a calculator.

   (a) $f(x) = \frac{\pi}{2} - \arccos\left(\frac{x}{5}\right)$  
   (b) $f(x) = 3 \arctan(4x)$.  
   (c) $f(x) = \arccot\left(\frac{x}{2}\right) + \pi$

**Solution.**

1. (a) Since $2 > 0$, we can use the property listed in Theorem 10.27 to rewrite $\arccot(2)$ as $\arccot(2) = \arctan\left(\frac{1}{2}\right)$. In ‘radian’ mode, we find $\arccot(2) = \arctan\left(\frac{1}{2}\right) \approx 0.4636$.

   (b) Since $5 \geq 1$, we can use the property from either Theorem 10.28 or Theorem 10.29 to write $\text{arcsec}(5) = \arccos\left(\frac{1}{5}\right) \approx 1.3694$. 
(c) Since the argument $-2$ is negative, we cannot directly apply Theorem 10.27 to help us find $\arccot(-2)$. Let $t = \arccot(-2)$. Then $t$ is a real number such that $0 < t < \pi$ and $\cot(t) = -2$. Moreover, since $\cot(t) < 0$, we know $\frac{\pi}{2} < t < \pi$. Geometrically, this means $t$ corresponds to a Quadrant II angle $\theta = t$ radians. This allows us to proceed using a 'reference angle' approach. Consider $\alpha$, the reference angle for $\theta$, as pictured below. By definition, $\alpha$ is an acute angle so $0 < \alpha < \frac{\pi}{2}$, and the Reference Angle Theorem, Theorem 10.2, tells us that $\cot(\alpha) = 2$. This means $\alpha = \arccot(2)$ radians. Since the argument of arccotangent is now a positive $2$, we can use Theorem 10.27 to get $\alpha = \arccot(2) = \arctan\left(\frac{1}{2}\right)$ radians. Since $\theta = \pi - \alpha = \pi - \arctan\left(\frac{1}{2}\right) \approx 2.6779$ radians, we get $\arccot(-2) \approx 2.6779$.

Another way to attack the problem is to use $\arctan\left(\frac{-1}{2}\right)$. By definition, the real number $t = \arctan\left(\frac{-1}{2}\right)$ satisfies $\tan(t) = -\frac{1}{2}$ with $-\frac{\pi}{2} < t < \frac{\pi}{2}$. Since $\tan(t) < 0$, we know more specifically that $-\frac{\pi}{2} < t < 0$, so $t$ corresponds to an angle $\beta$ in Quadrant IV. To find the value of $\arccot(-2)$, we once again visualize the angle $\theta = \arccot(-2)$ radians and note that it is a Quadrant II angle with $\tan(\theta) = -\frac{1}{2}$. This means it is exactly $\pi$ units away from $\beta$, and we get $\theta = \pi + \beta = \pi + \arctan\left(-\frac{1}{2}\right) \approx 2.6779$ radians. Hence, as before, $\arccot(-2) \approx 2.6779$. 
10.6 The Inverse Trigonometric Functions

(d) If the range of arccosecant is taken to be \([-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})\), we can use Theorem 10.28 to get \(\arccsc \left(-\frac{3}{2}\right) = \arcsin \left(-\frac{2}{3}\right) \approx -0.7297\). If, on the other hand, the range of arccosecant is taken to be \((0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})\), then we proceed as in the previous problem by letting \(t = \arccsc \left(-\frac{3}{2}\right)\). Then \(t\) is a real number with \(\csc(t) = -\frac{3}{2}\). Since \(\csc(t) < 0\), we have that \(\pi < \theta \leq \frac{3\pi}{2}\), so \(t\) corresponds to a Quadrant III angle, \(\theta\). As above, we let \(\alpha\) be the reference angle for \(\theta\). Then \(0 < \alpha < \frac{\pi}{2}\) and \(\csc(\alpha) = \frac{3}{2}\), which means \(\alpha = \arccsc \left(\frac{3}{2}\right)\) radians. Since the argument of arccosecant is now positive, we may use Theorem 10.29 to get \(\alpha = \arccsc \left(\frac{3}{2}\right) = \arcsin \left(\frac{2}{3}\right)\) radians. Since \(\theta = \pi + \alpha = \pi + \arcsin \left(\frac{2}{3}\right) \approx 3.8713\) radians, \(\arccsc \left(-\frac{3}{2}\right) \approx 3.8713\).
2. (a) Since the domain of \( F(x) = \arccos(x) \) is \(-1 \leq x \leq 1\), we can find the domain of
\[ f(x) = \frac{\pi}{2} - \arccos \left( \frac{x}{5} \right) \]
by setting the argument of the arccosine, in this case \( \frac{x}{5} \), between
\(-1\) and 1. Solving \(-1 \leq \frac{x}{5} \leq 1\) gives \(-5 \leq x \leq 5\), so the domain is \([-5, 5]\). To determine
the range of \( f \), we take a cue from Section 1.7. Three ‘key’ points on the graph of
\( F(x) = \arccos(x) \) are \((-1, \pi)\), \( (0, \frac{\pi}{2}) \) and \((1, 0)\). Following the procedure outlined in
Theorem 1.7, we track these points to \((-5, -\frac{\pi}{2})\), \((0, 0)\) and \((5, \frac{\pi}{2})\). Plotting these values
tells us that the range\(^5\) of \( f \) is \([-\frac{\pi}{2}, \frac{\pi}{2}]\). Our graph confirms our results.

(b) To find the domain and range of \( f(x) = 3\arctan(4x) \), we note that since the domain
of \( F(x) = \arctan(x) \) is all real numbers, the only restrictions, if any, on the domain of
\( f(x) = 3\arctan(4x) \) come from the argument of the arctangent, in this case, \( 4x \). Since
\( 4x \) is defined for all real numbers, we have established that the domain of \( f \) is all real
numbers. To determine the range of \( f \), we can, once again, appeal to Theorem 1.7.
Choosing our ‘key’ point to be \((0,0)\) and tracking the horizontal asymptotes \( y = -\frac{\pi}{2} \)
and \( y = \frac{\pi}{2} \), we find that the graph of \( y = f(x) = 3\arctan(4x) \) differs from the graph of
\( y = F(x) = \arctan(x) \) by a horizontal compression by a factor of 4 and a vertical stretch
by a factor of 3. It is the latter which affects the range, producing a range of \((-\frac{3\pi}{2}, \frac{3\pi}{2})\).
We confirm our findings on the calculator below.

\[ y = f(x) = \frac{\pi}{2} - \arccos \left( \frac{x}{5} \right) \quad \text{and} \quad y = f(x) = 3\arctan(4x) \]

(c) To find the domain of \( g(x) = \arccot \left( \frac{x}{5} \right) + \pi \), we proceed as above. Since the domain
of \( G(x) = \arccot(x) \) is \((-\infty, \infty)\), and \( \frac{x}{5} \) is defined for all \( x \), we get that the domain of \( g \) is
\((-\infty, \infty)\) as well. As for the range, we note that the range of \( G(x) = \arccot(x) \), like that
of \( F(x) = \arctan(x) \), is limited by a pair of horizontal asymptotes, in this case \( y = 0 \)
and \( y = \pi \). Following Theorem 1.7, we graph \( y = g(x) = \arccot \left( \frac{x}{5} \right) + \pi \) starting with
\( y = G(x) = \arccot(x) \) and first performing a horizontal expansion by a factor of 2 and
following that with a vertical shift upwards by \( \pi \). This latter transformation is the one
which affects the range, making it now \((\pi, 2\pi)\). To check this graphically, we encounter
a bit of a problem, since on many calculators, there is no shortcut button corresponding
to the arccotangent function. Taking a cue from number 1c, we attempt to rewrite
\( g(x) = \arccot \left( \frac{x}{5} \right) + \pi \) in terms of the arctangent function. Using Theorem 10.27, we have
that \( \arccot \left( \frac{x}{5} \right) = \arctan \left( \frac{5}{x} \right) \) when \( \frac{x}{5} > 0 \), or, in this case, when \( x > 0 \). Hence, for \( x > 0 \),
we have \( g(x) = \arctan \left( \frac{5}{x} \right) + \pi \). When \( \frac{x}{5} < 0 \), we can use the same argument in number
1c that gave us \( \arccot(-2) = \pi + \arctan(-\frac{1}{2}) \) to give us \( \arccot(\frac{x}{5}) = \pi + \arctan(\frac{5}{x}) \).

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\(^5\)It also confirms our domain!
Hence, for \( x < 0 \), \( g(x) = \pi + \arctan \left( \frac{x}{2} \right) + \pi = \arctan \left( \frac{x}{2} \right) + 2\pi \). What about \( x = 0 \)? We know \( g(0) = \arccot(0) + \pi = \pi \), and neither of the formulas for \( g \) involving \( \arctangent \) will produce this result.\(^6\) Hence, in order to graph \( y = g(x) \) on our calculators, we need to write it as a piecewise defined function:

\[
g(x) = \arccot \left( \frac{x}{2} \right) + \pi = \begin{cases} 
\arctan \left( \frac{2}{x} \right) + 2\pi, & \text{if } x < 0 \\
\pi, & \text{if } x = 0 \\
\arctan \left( \frac{2}{x} \right) + \pi, & \text{if } x > 0 
\end{cases}
\]

We show the input and the result below.

The inverse trigonometric functions are typically found in applications whenever the measure of an angle is required. One such scenario is presented in the following example.

**Example 10.6.6.**\(^7\) The roof on the house below has a ‘6/12 pitch’. This means that when viewed from the side, the roof line has a rise of 6 feet over a run of 12 feet. Find the angle of inclination from the bottom of the roof to the top of the roof. Express your answer in decimal degrees, rounded to the nearest hundredth of a degree.

**Solution.** If we divide the side view of the house down the middle, we find that the roof line forms the hypotenuse of a right triangle with legs of length 6 feet and 12 feet. Using Theorem 10.10, we

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\(^6\)Without Calculus, of course . . .

\(^7\)The authors would like to thank Dan Stitz for this problem and associated graphics.
find the angle of inclination, labeled \( \theta \) below, satisfies \( \tan(\theta) = \frac{6}{12} = \frac{1}{2} \). Since \( \theta \) is an acute angle, we can use the arctangent function and we find \( \theta = \arctan \left( \frac{1}{2} \right) \) radians \( \approx 26.56^\circ \).

10.6.4 Solving Equations Using the Inverse Trigonometric Functions.

In Sections 10.2 and 10.3, we learned how to solve equations like \( \sin(\theta) = \frac{1}{2} \) for angles \( \theta \) and \( \tan(t) = -1 \) for real numbers \( t \). In each case, we ultimately appealed to the Unit Circle and relied on the fact that the answers corresponded to a set of ‘common angles’ listed on page 724. If, on the other hand, we had been asked to find all angles with \( \sin(\theta) = \frac{1}{3} \) or solve \( \tan(t) = -2 \) for real numbers \( t \), we would have been hard-pressed to do so. With the introduction of the inverse trigonometric functions, however, we are now in a position to solve these equations. A good parallel to keep in mind is how the square root function can be used to solve certain quadratic equations. The equation \( x^2 = 4 \) is a lot like \( \sin(\theta) = \frac{1}{2} \) in that it has friendly, ‘common value’ answers \( x = \pm 2 \). The equation \( x^2 = 7 \), on the other hand, is a lot like \( \sin(\theta) = \frac{1}{3} \). We know\(^8\) there are answers, but we can’t express them using ‘friendly’ numbers.\(^9\) To solve \( x^2 = 7 \), we make use of the square root function and write \( x = \pm \sqrt{7} \). We can certainly approximate these answers using a calculator, but as far as exact answers go, we leave them as \( x = \pm \sqrt{7} \). In the same way, we will use the arcsine function to solve \( \sin(\theta) = \frac{1}{3} \), as seen in the following example.

Example 10.6.7. Solve the following equations.

1. Find all angles \( \theta \) for which \( \sin(\theta) = \frac{1}{3} \).
2. Find all real numbers \( t \) for which \( \tan(t) = -2 \).
3. Solve \( \sec(x) = -\frac{5}{3} \) for \( x \).

Solution.

1. If \( \sin(\theta) = \frac{1}{3} \), then the terminal side of \( \theta \), when plotted in standard position, intersects the Unit Circle at \( y = \frac{1}{3} \). Geometrically, we see that this happens at two places: in Quadrant I and Quadrant II. If we let \( \alpha \) denote the acute solution to the equation, then all the solutions

\(^8\) How do we know this again?\(^9\) This is all, of course, a matter of opinion. For the record, the authors find \( \pm \sqrt{7} \) just as ‘nice’ as \( \pm 2 \).
to this equation in Quadrant I are coterminal with $\alpha$, and $\alpha$ serves as the reference angle for all of the solutions to this equation in Quadrant II.

Since $\frac{1}{3}$ isn’t the sine of any of the ‘common angles’ discussed earlier, we use the arcsine functions to express our answers. The real number $t = \arcsin \left( \frac{1}{3} \right)$ is defined so it satisfies $0 < t < \frac{\pi}{2}$ with $\sin(t) = \frac{1}{3}$. Hence, $\alpha = \arcsin \left( \frac{1}{3} \right)$ radians. Since the solutions in Quadrant I are all coterminal with $\alpha$, we get part of our solution to be $\theta = \alpha + 2\pi k = \arcsin \left( \frac{1}{3} \right) + 2\pi k$ for integers $k$. Turning our attention to Quadrant II, we get one solution to be $\pi - \alpha$. Hence, the Quadrant II solutions are $\theta = \pi - \alpha + 2\pi k = \pi - \arcsin \left( \frac{1}{3} \right) + 2\pi k$, for integers $k$.

2. We may visualize the solutions to $\tan(t) = -2$ as angles $\theta$ with $\tan(\theta) = -2$. Since tangent is negative only in Quadrants II and IV, we focus our efforts there.

Since $-2$ isn’t the tangent of any of the ‘common angles’, we need to use the arctangent function to express our answers. The real number $t = \arctan(-2)$ satisfies $\tan(t) = -2$ and $-\frac{\pi}{2} < t < 0$. If we let $\beta = \arctan(-2)$ radians, we see that all of the Quadrant IV solutions
to \( \tan(\theta) = -2 \) are coterminal with \( \beta \). Moreover, the solutions from Quadrant II differ by exactly \( \pi \) units from the solutions in Quadrant IV, so all the solutions to \( \tan(\theta) = -2 \) are of the form \( \theta = \beta + \pi k = \arctan(-2) + \pi k \) for some integer \( k \). Switching back to the variable \( t \), we record our final answer to \( \tan(t) = -2 \) as \( t = \arctan(-2) + \pi k \) for integers \( k \).

3. The last equation we are asked to solve, \( \sec(x) = -\frac{5}{3} \), poses two immediate problems. First, we are not told whether or not \( x \) represents an angle or a real number. We assume the latter, but note that we will use angles and the Unit Circle to solve the equation regardless. Second, as we have mentioned, there is no universally accepted range of the arcsecant function. For that reason, we adopt the advice given in Section 10.3 and convert this to the cosine problem \( \cos(x) = -\frac{3}{5} \). Adopting an angle approach, we consider the equation \( \cos(\theta) = -\frac{3}{5} \) and note that our solutions lie in Quadrants II and III. Since \( -\frac{3}{5} \) isn’t the cosine of any of the ‘common angles’, we’ll need to express our solutions in terms of the arccosine function. The real number \( t = \arccos\left(-\frac{3}{5}\right) \) is defined so that \( \frac{\pi}{2} < t < \pi \) with \( \cos(t) = -\frac{3}{5} \). If we let \( \beta = \arccos\left(-\frac{3}{5}\right) \) radians, we see that \( \beta \) is a Quadrant II angle. To obtain a Quadrant III angle solution, we may simply use \( -\beta = -\arccos\left(-\frac{3}{5}\right) \). Since all angle solutions are coterminal with \( \beta \) or \( -\beta \), we get our solutions to \( \cos(\theta) = -\frac{3}{5} \) to be \( \theta = \beta + 2\pi k = \arccos\left(-\frac{3}{5}\right) + 2\pi k \) or \( \theta = -\beta + 2\pi k = -\arccos\left(-\frac{3}{5}\right) + 2\pi k \) for integers \( k \). Switching back to the variable \( x \), we record our final answer to \( \sec(x) = -\frac{5}{3} \) as \( x = \arccos\left(-\frac{3}{5}\right) + 2\pi k \) or \( x = -\arccos\left(-\frac{3}{5}\right) + 2\pi k \) for integers \( k \).

The reader is encouraged to check the answers found in Example 10.6.7 - both analytically and with the calculator (see Section 10.6.3). With practice, the inverse trigonometric functions will become as familiar to you as the square root function. Speaking of practice . . .
10.7 Trigonometric Equations and Inequalities

In Sections 10.2, 10.3 and most recently 10.6, we solved some basic equations involving the trigonometric functions. Below we summarize the techniques we’ve employed thus far. Note that we use the neutral letter ‘\( u \)’ as the argument\(^1\) of each circular function for generality.

### Strategies for Solving Basic Equations Involving Trigonometric Functions

- To solve \( \cos(u) = c \) or \( \sin(u) = c \) for \(-1 \leq c \leq 1\), first solve for \( u \) in the interval \([0, 2\pi)\) and add integer multiples of the period \(2\pi\). If \(c < -1\) or \(c > 1\), there are no real solutions.

- To solve \( \sec(u) = c \) or \( \csc(u) = c \) for \(c \leq -1\) or \(c \geq 1\), convert to cosine or sine, respectively, and solve as above. If \(-1 < c < 1\), there are no real solutions.

- To solve \( \tan(u) = c \) for any real number \(c\), first solve for \( u \) in the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\) and add integer multiples of the period \(\pi\).

- To solve \( \cot(u) = c \) for \(c \neq 0\), convert to tangent and solve as above. If \(c = 0\), the solution to \( \cot(u) = 0 \) is \( u = \frac{\pi}{2} + \pi k \) for integers \(k\).

Using the above guidelines, we can comfortably solve \( \sin(x) = \frac{1}{2} \) and find the solution \( x = \frac{\pi}{6} + 2\pi k \) or \( x = \frac{5\pi}{6} + 2\pi k \) for integers \(k\). How do we solve something like \( \sin(3x) = \frac{1}{2} \)? Since this equation has the form \( \sin(u) = \frac{1}{2} \), we know the solutions take the form \( u = \frac{\pi}{6} + 2\pi k \) or \( u = \frac{5\pi}{6} + 2\pi k \) for integers \(k\). Since the argument of sine here is \(3x\), we have \(3x = \frac{\pi}{6} + 2\pi k \) or \(3x = \frac{5\pi}{6} + 2\pi k \) for integers \(k\). To solve for \(x\), we divide both sides\(^2\) of these equations by 3, and obtain \( x = \frac{\pi}{18} + \frac{2\pi}{3} k \) or \( x = \frac{5\pi}{18} + \frac{2\pi}{3} k \) for integers \(k\). This is the technique employed in the example below.

### Example 10.7.1.

Solve the following equations and check your answers analytically. List the solutions which lie in the interval \([0, 2\pi)\) and verify them using a graphing utility.

1. \( \cos(2x) = -\frac{\sqrt{3}}{2} \)
2. \( \csc\left(\frac{1}{3}x - \pi\right) = \sqrt{2} \)
3. \( \cot(3x) = 0 \)
4. \( \sec^2(x) = 4 \)
5. \( \tan\left(\frac{\pi}{2}\right) = -3 \)
6. \( \sin(2x) = 0.87 \)

**Solution.**

1. The solutions to \( \cos(u) = -\frac{\sqrt{3}}{2} \) are \( u = \frac{5\pi}{6} + 2\pi k \) or \( u = \frac{7\pi}{6} + 2\pi k \) for integers \(k\). Since the argument of cosine here is \(2x\), this means \(2x = \frac{5\pi}{6} + 2\pi k \) or \(2x = \frac{7\pi}{6} + 2\pi k \) for integers \(k\). Solving for \(x\) gives \( x = \frac{5\pi}{12} + \pi k \) or \( x = \frac{7\pi}{12} + \pi k \) for integers \(k\). To check these answers analytically, we substitute them into the original equation. For any integer \(k\) we have

   \[
   \cos\left(2\left[\frac{5\pi}{12} + \pi k\right]\right) = \cos\left(\frac{5\pi}{6} + 2\pi k\right) = \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}
   \]

   (the period of cosine is \(2\pi\))

---

\(^1\)See the comments at the beginning of Section 10.5 for a review of this concept.

\(^2\)Don’t forget to divide the \(2\pi k\) by 3 as well!
Similarly, we find \( \cos \left[ 2 \left( \frac{7\pi}{12} + \pi k \right) \right] = \cos \left( \frac{7\pi}{6} + 2\pi k \right) = \cos \left( \frac{7\pi}{6} \right) = -\frac{\sqrt{3}}{2} \). To determine which of our solutions lie in \([0, 2\pi]\), we substitute integer values for \( k \). The solutions we keep come from the values of \( k = 0 \) and \( k = 1 \) and are \( x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12} \) and \( \frac{19\pi}{12} \). Using a calculator, we graph \( y = \cos(2x) \) and \( y = -\frac{\sqrt{3}}{2} \) over \([0, 2\pi]\) and examine where these two graphs intersect. We see that the \( x \)-coordinates of the intersection points correspond to the decimal representations of our exact answers.

2. Since this equation has the form \( \csc(u) = \sqrt{2} \), we rewrite this as \( \sin(u) = \frac{\sqrt{2}}{2} \) and find \( u = \frac{\pi}{4} + 2\pi k \) or \( u = \frac{3\pi}{4} + 2\pi k \) for integers \( k \). Since the argument of cosecant here is \( \left( \frac{1}{3}x - \pi \right) \),

\[
\frac{1}{3}x - \pi = \frac{\pi}{4} + 2\pi k \quad \text{or} \quad \frac{1}{3}x - \pi = \frac{3\pi}{4} + 2\pi k
\]

To solve \( \frac{1}{3}x - \pi = \frac{\pi}{4} + 2\pi k \), we first add \( \pi \) to both sides

\[
\frac{1}{3}x = \frac{\pi}{4} + 2\pi k + \pi
\]

A common error is to treat the ‘2\(\pi k\)’ and ‘\(\pi\)’ terms as ‘like’ terms and try to combine them when they are not.\(^3\) We can, however, combine the ‘\(\pi\)’ and ‘\(\frac{\pi}{4}\)’ terms to get

\[
\frac{1}{3}x = \frac{5\pi}{4} + 2\pi k
\]

We now finish by multiplying both sides by 3 to get

\[
x = 3 \left( \frac{5\pi}{4} + 2\pi k \right) = \frac{15\pi}{4} + 6\pi k
\]

Solving the other equation, \( \frac{1}{3}x - \pi = \frac{3\pi}{4} + 2\pi k \) produces \( x = \frac{21\pi}{4} + 6\pi k \) for integers \( k \). To check the first family of answers, we substitute, combine line terms, and simplify.

\[
csc \left( \frac{1}{3} \left[ \frac{15\pi}{4} + 6\pi k \right] - \pi \right) = \csc \left( \frac{5\pi}{4} + 2\pi k - \pi \right) = \csc \left( \frac{\pi}{4} + 2\pi k \right) = \csc \left( \frac{\pi}{4} \right) = \sqrt{2} \quad \text{(the period of cosecant is } 2\pi)\]

The family \( x = \frac{21\pi}{4} + 6\pi k \) checks similarly. Despite having infinitely many solutions, we find that none of them lie in \([0, 2\pi]\). To verify this graphically, we use a reciprocal identity to rewrite the cosecant as a sine and we find that \( y = \frac{1}{\sin \left( \frac{1}{3}x - \pi \right)} \) and \( y = \sqrt{2} \) do not intersect at all over the interval \([0, 2\pi]\).

\(^3\)Do you see why?
3. Since \( \cot(3x) = 0 \) has the form \( \cot(u) = 0 \), we know \( u = \frac{\pi}{2} + \pi k \), so, in this case, \( 3x = \frac{\pi}{6} + \frac{\pi}{3} k \). Solving for \( x \) yields \( x = \frac{\pi}{6} + \frac{\pi}{3} k \). Checking our answers, we get
\[
\cot \left( 3 \left[ \frac{\pi}{6} + \frac{\pi}{3} k \right] \right) = \cot \left( \frac{\pi}{2} + \pi k \right) = \cot \left( \frac{\pi}{2} \right) \quad \text{(the period of cotangent is } \pi) \]
\[
= 0
\]
As \( k \) runs through the integers, we obtain six answers, corresponding to \( k = 0 \) through \( k = 5 \), which lie in \([0, 2\pi)\): \( x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2} \) and \( \frac{11\pi}{6} \). To confirm these graphically, we must be careful. On many calculators, there is no function button for cotangent. We choose \( 4 \) to use the quotient identity \( \cot(3x) = \frac{\cos(3x)}{\sin(3x)} \). Graphing \( y = \cos(3x) \sin(3x) \) and \( y = 0 \) (the \( x \)-axis), we see that the \( x \)-coordinates of the intersection points approximately match our solutions.

4. The complication in solving an equation like \( \sec^2(x) = 4 \) comes not from the argument of secant, which is just \( x \), but rather, the fact the secant is being squared. To get this equation to look like one of the forms listed on page 857, we extract square roots to get \( \sec(x) = \pm 2 \). Converting to cosines, we have \( \cos(x) = \pm \frac{1}{2} \). For \( \cos(x) = 1/2 \), we get \( x = \frac{\pi}{3} + 2\pi k \) or \( x = \frac{5\pi}{3} + 2\pi k \) for integers \( k \). For \( \cos(x) = -1/2 \), we get \( x = \frac{2\pi}{3} + 2\pi k \) or \( x = \frac{4\pi}{3} + 2\pi k \) for integers \( k \). If we take a step back and think of these families of solutions geometrically, we see we are finding the measures of all angles with a reference angle of \( \frac{\pi}{3} \). As a result, these solutions can be combined and we may write our solutions as \( x = \frac{\pi}{3} + \pi k \) and \( x = \frac{2\pi}{3} + \pi k \) for integers \( k \). To check the first family of solutions, we note that, depending on the integer \( k \), \( \sec \left( \frac{\pi}{3} + \pi k \right) \) doesn’t always equal \( \sec \left( \frac{\pi}{3} \right) \). However, it is true that for all integers \( k \), \( \sec \left( \frac{\pi}{3} + \pi k \right) = \pm \sec \left( \frac{\pi}{3} \right) = \pm 2 \). (Can you show this?) As a result,
\[
\sec^2 \left( \frac{\pi}{3} + \pi k \right) = \left( \pm \sec \left( \frac{\pi}{3} \right) \right)^2 = (\pm 2)^2 = 4
\]
The same holds for the family \( x = \frac{2\pi}{3} + \pi k \). The solutions which lie in \([0, 2\pi)\) come from the values \( k = 0 \) and \( k = 1 \), namely \( x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3} \) and \( \frac{5\pi}{3} \). To confirm graphically, we use

---

4The reader is encouraged to see what happens if we had chosen the reciprocal identity \( \cot(3x) = \frac{1}{\tan(3x)} \) instead. The graph on the calculator appears identical, but what happens when you try to find the intersection points?
a reciprocal identity to rewrite the secant as cosine. The $x$-coordinates of the intersection points of $y = \frac{1}{\cos(x)^2}$ and $y = 4$ verify our answers.

5. The equation $\tan(\frac{x}{2}) = -3$ has the form $\tan(u) = -3$, whose solution is $u = \arctan(-3) + \pi k$. Hence, $\frac{x}{2} = \arctan(-3) + \pi k$, so $x = 2 \arctan(-3) + 2\pi k$ for integers $k$. To check, we note

$$\tan\left(\frac{2\arctan(-3) + 2\pi k}{2}\right) = \tan\left(\arctan(-3) + \pi k\right) = \tan\left(\arctan(-3)\right) \quad \text{(the period of tangent is $\pi$)}$$

$$= -3 \quad \text{(See Theorem 10.27)}$$

To determine which of our answers lie in the interval $[0, 2\pi)$, we first need to get an idea of the value of $2 \arctan(-3)$. While we could easily find an approximation using a calculator, we proceed analytically. Since $-3 < 0$, it follows that $-\frac{\pi}{2} < \arctan(-3) < 0$. Multiplying through by 2 gives $-\pi < 2 \arctan(-3) < 0$. We are now in a position to argue which of the solutions $x = 2 \arctan(-3) + 2\pi k$ lie in $[0, 2\pi)$. For $k = 0$, we get $x = 2 \arctan(-3) < 0$, so we discard this answer and all answers $x = 2 \arctan(-3) + 2\pi k$ where $k < 0$. Next, we turn our attention to $k = 1$ and get $x = 2 \arctan(-3) + 2\pi$. Starting with the inequality $-\pi < 2 \arctan(-3) < 0$, we add $2\pi$ and get $\pi < 2 \arctan(-3) + 2\pi < 2\pi$. This means $x = 2 \arctan(-3) + 2\pi$ lies in $[0, 2\pi)$. Advancing $k$ to 2 produces $x = 2 \arctan(-3) + 4\pi$. Once again, we get from $-\pi < 2 \arctan(-3) < 0$ that $3\pi < 2 \arctan(-3) + 4\pi < 4\pi$. Since this is outside the interval $[0, 2\pi)$, we discard $x = 2 \arctan(-3) + 4\pi$ and all solutions of the form $x = 2 \arctan(-3) + 2\pi k$ for $k > 2$. Graphically, we see $y = \tan\left(\frac{x}{2}\right)$ and $y = -3$ intersect only once on $[0, 2\pi)$ at $x = 2 \arctan(-3) + 2\pi \approx 3.7851$.

6. To solve $\sin(2x) = 0.87$, we first note that it has the form $\sin(u) = 0.87$, which has the family of solutions $u = \arcsin(0.87) + 2\pi k$ or $u = \pi - \arcsin(0.87) + 2\pi k$ for integers $k$. Since the argument of sine here is $2x$, we get $2x = \arcsin(0.87) + 2\pi k$ or $2x = \pi - \arcsin(0.87) + 2\pi k$ which gives $x = \frac{1}{2} \arcsin(0.87) + \pi k$ or $x = \frac{\pi}{2} - \frac{1}{2} \arcsin(0.87) + \pi k$ for integers $k$. To check,

5Your instructor will let you know if you should abandon the analytic route at this point and use your calculator. But seriously, what fun would that be?
\[
\sin \left( 2 \left[ \frac{1}{2} \arcsin(0.87) + \pi k \right] \right) = \sin \left( \arcsin(0.87) + 2\pi k \right)
\]
\[
= \sin \left( \arcsin(0.87) \right) \quad \text{(the period of sine is } 2\pi)\]
\[
= 0.87 \quad \text{(See Theorem 10.26)}
\]

For the family \( x = \frac{\pi}{2} - \frac{1}{2} \arcsin(0.87) + \pi k \), we get

\[
\sin \left( 2 \left[ \frac{\pi}{2} - \frac{1}{2} \arcsin(0.87) + \pi k \right] \right) = \sin \left( \pi - \arcsin(0.87) + 2\pi k \right)
\]
\[
= \sin \left( \arcsin(0.87) \right) \quad \text{(the period of sine is } 2\pi)\]
\[
= 0.87 \quad \text{(See Theorem 10.26)}
\]

To determine which of these solutions lie in \([0, 2\pi]\), we first need to get an idea of the value of \( x = \frac{1}{2} \arcsin(0.87) \). Once again, we could use the calculator, but we adopt an analytic route here. By definition, \( 0 < \arcsin(0.87) < \frac{\pi}{2} \) so that multiplying through by \( \frac{1}{2} \) gives us \( 0 < \frac{1}{2} \arcsin(0.87) < \frac{\pi}{4} \). Starting with the family of solutions \( x = \frac{1}{2} \arcsin(0.87) + \pi k \), we use the same kind of arguments as in our solution to number 5 above and find only the solutions corresponding to \( k = 0 \) and \( k = 1 \) lie in \([0, 2\pi]\): \( x = \frac{1}{2} \arcsin(0.87) \) and \( x = \frac{1}{2} \arcsin(0.87) + \pi \).

Next, we move to the family \( x = \frac{\pi}{2} - \frac{1}{2} \arcsin(0.87) + \pi k \) for integers \( k \). Here, we need to get a better estimate of \( \frac{\pi}{2} - \frac{1}{2} \arcsin(0.87) \). From the inequality \( 0 < \frac{1}{2} \arcsin(0.87) < \frac{\pi}{4} \), we first multiply through by \(-1\) and then add \( \frac{\pi}{2} \) to get \( \frac{\pi}{2} > \frac{\pi}{2} - \frac{1}{2} \arcsin(0.87) > \frac{\pi}{4} \), or \( \frac{\pi}{4} < \frac{\pi}{2} - \frac{1}{2} \arcsin(0.87) < \frac{\pi}{2} \). Proceeding with the usual arguments, we find the only solutions which lie in \([0, 2\pi]\) correspond to \( k = 0 \) and \( k = 1 \), namely \( x = \frac{\pi}{2} - \frac{1}{2} \arcsin(0.87) \) and \( x = \frac{3\pi}{2} - \frac{1}{2} \arcsin(0.87) \). All told, we have found four solutions to \( \sin(2x) = 0.87 \) in \([0, 2\pi]\): \( x = \frac{1}{2} \arcsin(0.87), x = \frac{1}{2} \arcsin(0.87) + \pi, x = \frac{\pi}{2} - \frac{1}{2} \arcsin(0.87) \) and \( x = \frac{3\pi}{2} - \frac{1}{2} \arcsin(0.87) \).

By graphing \( y = \sin(2x) \) and \( y = 0.87 \), we confirm our results.

\[\text{Graphs: } \frac{\pi}{2} \quad \text{and} \quad \frac{3\pi}{2} \]

\[\text{Graphs: } \frac{\pi}{2} \quad \text{and} \quad \frac{3\pi}{2} \]

\[y = \tan \left( \frac{x}{2} \right) \quad \text{and} \quad y = -3\]

\[y = \sin(2x) \quad \text{and} \quad y = 0.87\]
Each of the problems in Example 10.7.1 featured one trigonometric function. If an equation involves two different trigonometric functions or if the equation contains the same trigonometric function but with different arguments, we will need to use identities and Algebra to reduce the equation to the same form as those given on page 857.

Example 10.7.2. Solve the following equations and list the solutions which lie in the interval \([0, 2\pi]\). Verify your solutions on \([0, 2\pi]\) graphically.

1. \(3 \sin^3(x) = \sin^2(x)\)
2. \(\sec^2(x) = \tan(x) + 3\)
3. \(\cos(2x) = 3 \cos(x) - 2\)
4. \(\cos(3x) = 2 - \cos(x)\)
5. \(\cos(3x) = \cos(5x)\)
6. \(\sin(2x) = \sqrt{3} \cos(x)\)
7. \(\sin(x) \cos(\frac{x}{2}) + \cos(x) \sin(\frac{x}{2}) = 1\)
8. \(\cos(x) - \sqrt{3} \sin(x) = 2\)

Solution.

1. We resist the temptation to divide both sides of \(3 \sin^3(x) = \sin^2(x)\) by \(\sin^2(x)\) (What goes wrong if you do?) and instead gather all of the terms to one side of the equation and factor.

\[
3 \sin^3(x) - \sin^2(x) = 0
\]

Factoring out \(\sin^2(x)\) from both terms,

\[
\sin^2(x)(3 \sin(x) - 1) = 0
\]

We get \(\sin^2(x) = 0\) or \(3 \sin(x) - 1 = 0\). Solving for \(\sin(x)\), we find \(\sin(x) = 0\) or \(\sin(x) = \frac{1}{3}\). The solution to the first equation is \(x = \pi k\), with \(x = 0\) and \(x = \pi\) being the two solutions which lie in \([0, 2\pi]\). To solve \(\sin(x) = \frac{1}{3}\), we use the arcsine function to get \(x = \arcsin\left(\frac{1}{3}\right) + 2\pi k\) or \(x = \pi - \arcsin\left(\frac{1}{3}\right) + 2\pi k\) for integers \(k\). We find the two solutions here which lie in \([0, 2\pi]\) to be \(x = \arcsin\left(\frac{1}{3}\right)\) and \(x = \pi - \arcsin\left(\frac{1}{3}\right)\). To check graphically, we plot \(y = (\sin(x))^2\) and find the \(x\)-coordinates of the intersection points of these two curves. Some extra zooming is required near \(x = 0\) and \(x = \pi\) to verify that these two curves do in fact intersect four times.

2. Analysis of \(\sec^2(x) = \tan(x) + 3\) reveals two different trigonometric functions, so an identity is in order. Since \(\sec^2(x) = 1 + \tan^2(x)\), we get

\[
\begin{align*}
\sec^2(x) &= \tan(x) + 3 \\
1 + \tan^2(x) &= \tan(x) + 3 \\
\tan^2(x) - \tan(x) - 2 &= 0 \\
u^2 - u - 2 &= 0 \\
(u + 1)(u - 2) &= 0
\end{align*}
\]

Let \(u = \tan(x)\).

\[\text{Note that we are not counting the point } (2\pi, 0) \text{ in our solution set since } x = 2\pi \text{ is not in the interval } [0, 2\pi]. \text{ In the forthcoming solutions, remember that while } x = 2\pi \text{ may be a solution to the equation, it isn’t counted among the solutions in } [0, 2\pi].\]
This gives \( u = -1 \) or \( u = 2 \). Since \( u = \tan(x) \), we have \( \tan(x) = -1 \) or \( \tan(x) = 2 \). From \( \tan(x) = -1 \), we get \( x = -\frac{\pi}{4} + \pi k \) for integers \( k \). To solve \( \tan(x) = 2 \), we employ the arctangent function and get \( x = \arctan(2) + \pi k \) for integers \( k \). From the first set of solutions, we get \( x = \frac{3\pi}{4} \) and \( x = \frac{7\pi}{4} \) as our answers which lie in \([0, 2\pi)\). Using the same sort of argument we saw in Example 10.7.1, we get \( x = \arctan(2) \) and \( x = \pi + \arctan(2) \) as answers from our second set of solutions which lie in \([0, 2\pi)\).

Using a reciprocal identity, we rewrite the secant as a cosine and graph \( y = \frac{1}{\cos(x)} \) and \( y = \tan(x) + 3 \) to find the \( x \)-values of the points where they intersect.

3. In the equation \( \cos(2x) = 3\cos(x) - 2 \), we have the same circular function, namely cosine, on both sides but the arguments differ. Using the identity \( \cos(2x) = 2\cos^2(x) - 1 \), we obtain a ‘quadratic in disguise’ and proceed as we have done in the past.

\[
\begin{align*}
\cos(2x) &= 3\cos(x) - 2 \\
2\cos^2(x) - 1 &= 3\cos(x) - 2 \quad \text{(Since } \cos(2x) = 2\cos^2(x) - 1.)
\end{align*}
\]

\( 2\cos^2(x) - 3\cos(x) + 1 = 0 \)

\( 2u^2 - 3u + 1 = 0 \)

\( (2u - 1)(u - 1) = 0 \)

This gives \( u = \frac{1}{2} \) or \( u = 1 \). Since \( u = \cos(x) \), we get \( \cos(x) = \frac{1}{2} \) or \( \cos(x) = 1 \). Solving \( \cos(x) = \frac{1}{2} \), we get \( x = \frac{\pi}{3} + 2\pi k \) or \( x = \frac{5\pi}{3} + 2\pi k \) for integers \( k \). From \( \cos(x) = 1 \), we get \( x = 2\pi k \) for integers \( k \). The answers which lie in \([0, 2\pi)\) are \( x = 0, \frac{\pi}{3}, \text{ and } \frac{5\pi}{3} \). Graphing \( y = \cos(2x) \) and \( y = 3\cos(x) - 2 \), we find, after a little extra effort, that the curves intersect in three places on \([0, 2\pi)\), and the \( x \)-coordinates of these points confirm our results.

4. To solve \( \cos(3x) = 2 - \cos(x) \), we use the same technique as in the previous problem. From Example 10.4.3, number 4, we know that \( \cos(3x) = 4\cos^3(x) - 3\cos(x) \). This transforms the equation into a polynomial in terms of \( \cos(x) \).

\[
\begin{align*}
\cos(3x) &= 2 - \cos(x) \\
4\cos^3(x) - 3\cos(x) &= 2 - \cos(x) \\
2\cos^3(x) - 2\cos(x) - 2 &= 0 \\
4u^3 - 2u - 2 &= 0 \quad \text{Let } u = \cos(x).
\end{align*}
\]
To solve $4u^3 - 2u - 2 = 0$, we need the techniques in Chapter 3 to factor $4u^3 - 2u - 2$ into $(u - 1)(4u^2 + 4u + 2)$. We get either $u - 1 = 0$ or $4u^2 + 2u + 2 = 0$, and since the discriminant of the latter is negative, the only real solution to $4u^3 - 2u - 2 = 0$ is $u = 1$. Since $u = \cos(x)$, we get $\cos(x) = 1$, so $x = 2\pi k$ for integers $k$. The only solution which lies in $[0, 2\pi)$ is $x = 0$.

Graphing $y = \cos(3x)$ and $y = 3\cos(x) - 2$ on the same set of axes over $[0, 2\pi)$ shows that the graphs intersect at what appears to be $(0, 1)$, as required.

5. While we could approach $\cos(3x) = \cos(5x)$ in the same manner as we did the previous two problems, we choose instead to showcase the utility of the Sum to Product Identities. From $\cos(3x) = \cos(5x)$, we get $\cos(5x) - \cos(3x) = 0$, and it is the presence of 0 on the right hand side that indicates a switch to a product would be a good move. Using Theorem 10.21, we have that $\cos(5x) - \cos(3x) = -2 \sin \left( \frac{5x + 3x}{2} \right) \sin \left( \frac{5x - 3x}{2} \right) = -2 \sin(4x) \sin(x)$. Hence, the equation $\cos(5x) = \cos(3x)$ is equivalent to $-2 \sin(4x) \sin(x) = 0$. From this, we get $\sin(4x) = 0$ or $\sin(x) = 0$. Solving $\sin(4x) = 0$ gives $x = \frac{\pi}{4} k$ for integers $k$, and the solution to $\sin(x) = 0$ is $x = \pi k$ for integers $k$. The second set of solutions is contained in the first set of solutions, so our final solution to $\cos(5x) = \cos(3x)$ is $x = \frac{\pi}{4} k$ for integers $k$. There are eight of these answers which lie in $[0, 2\pi)$: $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}$ and $\frac{7\pi}{4}$. Our plot of the graphs of $y = \cos(3x)$ and $y = \cos(5x)$ below (after some careful zooming) bears this out.

6. In examining the equation $\sin(2x) = \sqrt{3} \cos(x)$, not only do we have different circular functions involved, namely sine and cosine, we also have different arguments to contend with, namely $2x$ and $x$. Using the identity $\sin(2x) = 2 \sin(x) \cos(x)$ makes all of the arguments the same and we proceed as we would solving any nonlinear equation – gather all of the nonzero terms on one side of the equation and factor.

\[
\sin(2x) = \sqrt{3} \cos(x) \\
2 \sin(x) \cos(x) = \sqrt{3} \cos(x) \quad \text{(Since \, \sin(2x) = 2 \sin(x) \cos(x).)}
\]

\[
2 \sin(x) \cos(x) - \sqrt{3} \cos(x) = 0 \\
\cos(x)(2 \sin(x) - \sqrt{3}) = 0
\]

from which we get $\cos(x) = 0$ or $\sin(x) = \frac{\sqrt{3}}{2}$. From $\cos(x) = 0$, we obtain $x = \frac{\pi}{2} + \pi k$ for integers $k$. From $\sin(x) = \frac{\sqrt{3}}{2}$, we get $x = \frac{\pi}{3} + 2\pi k$ or $x = \frac{2\pi}{3} + 2\pi k$ for integers $k$. The answers

---

\(^7\)As always, experience is the greatest teacher here!

\(^8\)As always, when in doubt, write it out!
which lie in \([0, 2\pi]\) are \(x = \frac{\pi}{2}, \frac{3\pi}{4}, \frac{\pi}{3}\) and \(\frac{2\pi}{3}\). We graph \(y = \sin(2x)\) and \(y = \sqrt{3}\cos(x)\) and, after some careful zooming, verify our answers.

7. Unlike the previous problem, there seems to be no quick way to get the circular functions or their arguments to match in the equation \(\sin(x) \cos\left(\frac{x}{2}\right) + \cos(x) \sin\left(\frac{x}{2}\right) = 1\). If we stare at it long enough, however, we realize that the left hand side is the expanded form of the sum formula for \(\sin\left(x + \frac{x}{2}\right)\). Hence, our original equation is equivalent to \(\sin\left(\frac{3x}{2}\right) = 1\). Solving, we find \(x = \frac{\pi}{3} + \frac{4\pi}{3}k\) for integers \(k\). Two of these solutions lie in \([0, 2\pi]\): \(x = \frac{\pi}{3}\) and \(x = \frac{5\pi}{3}\). Graphing \(y = \sin(x) \cos\left(\frac{x}{2}\right) + \cos(x) \sin\left(\frac{x}{2}\right)\) and \(y = 1\) validates our solutions.

8. With the absence of double angles or squares, there doesn’t seem to be much we can do. However, since the arguments of the cosine and sine are the same, we can rewrite the left hand side of this equation as a sinusoid.\(^9\) To fit \(f(x) = \cos(x) - \sqrt{3}\sin(x)\) to the form \(A \sin(\omega t + \phi) + B\), we use what we learned in Example 10.5.3 and find \(A = 2, B = 0, \omega = 1\) and \(\phi = \frac{5\pi}{6}\). Hence, we can rewrite the equation \(\cos(x) - \sqrt{3}\sin(x) = 2\) as \(2 \sin\left(x + \frac{5\pi}{6}\right) = 2\), or \(\sin\left(x + \frac{5\pi}{6}\right) = 1\). Solving the latter, we get \(x = -\frac{\pi}{3} + 2\pi k\) for integers \(k\). Only one of these solutions, \(x = \frac{5\pi}{3}\), which corresponds to \(k = 1\), lies in \([0, 2\pi]\). Geometrically, we see that \(y = \cos(x) - \sqrt{3}\sin(x)\) and \(y = 2\) intersect just once, supporting our answer.

---

\(^9\)We are essentially 'undoing' the sum / difference formula for cosine or sine, depending on which form we use, so this problem is actually closely related to the previous one!
Next, we focus on solving inequalities involving the trigonometric functions. Since these functions are continuous on their domains, we may use the sign diagram technique we’ve used in the past to solve the inequalities.\textsuperscript{10}

Example 10.7.3. Solve the following inequalities on \([0, 2\pi]\). Express your answers using interval notation and verify your answers graphically.

1. \(2 \sin(x) \leq 1\)
2. \(\sin(2x) > \cos(x)\)
3. \(\tan(x) \geq 3\)

Solution.

1. We begin solving \(2 \sin(x) \leq 1\) by collecting all of the terms on one side of the equation and zero on the other to get \(2 \sin(x) - 1 \leq 0\). Next, we let \(f(x) = 2 \sin(x) - 1\) and note that our original inequality is equivalent to solving \(f(x) \leq 0\). We now look to see where, if ever, \(f(x)\) is undefined and where \(f(x) = 0\). Since the domain of \(f(x)\) is all real numbers, we can immediately set about finding the zeros of \(f(x)\). Solving \(f(x) = 0\), we have \(2 \sin(x) - 1 = 0\) or \(\sin(x) = \frac{1}{2}\). The solutions here are \(x = \frac{\pi}{6} + 2\pi k\) and \(x = \frac{5\pi}{6} + 2\pi k\) for integers \(k\). Since we are restricting our attention to \([0, 2\pi]\), only \(x = \frac{\pi}{6}\) and \(x = \frac{5\pi}{6}\) are of concern to us. Next, we choose test values in \([0, 2\pi]\) other than the zeros and determine if \(f(x)\) is positive or negative there. For \(x = 0\) we have \(f(0) = -1\), for \(x = \frac{\pi}{2}\) we get \(f\left(\frac{\pi}{2}\right) = 1\) and for \(x = \pi\) we get \(f(\pi) = -1\). Since our original inequality is equivalent to \(f(x) \leq 0\), we are looking for where the function is negative (\(-\)) or 0, and we get the intervals \([0, \frac{\pi}{6}] \cup [\frac{5\pi}{6}, 2\pi]\). We can confirm our answer graphically by seeing where the graph of \(y = 2 \sin(x)\) crosses or is below the graph of \(y = 1\).

\[
\begin{array}{cccc}
(-) & 0 & (+) & 0 & (-) \\
0 & \frac{\pi}{6} & \frac{5\pi}{6} & 2\pi
\end{array}
\]

\(y = 2 \sin(x)\) and \(y = 1\)

2. We first rewrite \(\sin(2x) > \cos(x)\) as \(\sin(2x) - \cos(x) > 0\) and let \(f(x) = \sin(2x) - \cos(x)\). Our original inequality is thus equivalent to \(f(x) > 0\). The domain of \(f(x)\) is all real numbers, so we can advance to finding the zeros of \(f\). Setting \(f(x) = 0\) yields \(\sin(2x) - \cos(x) = 0\), which, by way of the double angle identity for sine, becomes \(2 \sin(x) \cos(x) - \cos(x) = 0\) or \(\cos(x)(2 \sin(x) - 1) = 0\). From \(\cos(x) = 0\), we get \(x = \frac{\pi}{2} + \pi k\) for integers \(k\) of which only \(x = \frac{\pi}{2}\) and \(x = \frac{3\pi}{2}\) lie in \([0, 2\pi]\). For \(2 \sin(x) - 1 = 0\), we get \(\sin(x) = \frac{1}{2}\) which gives \(x = \frac{\pi}{6} + 2\pi k\) or \(x = \frac{5\pi}{6} + 2\pi k\) for integers \(k\). Of those, only \(x = \frac{\pi}{6}\) and \(x = \frac{5\pi}{6}\) lie in \([0, 2\pi]\). Next, we choose

\textsuperscript{10}See page 214, Example 3.1.5, page 321, page 399, Example 6.3.2 and Example 6.4.2 for discussion of this technique.
our test values. For \( x = 0 \) we find \( f(0) = -1 \); when \( x = \frac{\pi}{4} \) we get \( f \left( \frac{\pi}{4} \right) = 1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2} \); for \( x = \frac{3\pi}{4} \) we get \( f \left( \frac{3\pi}{4} \right) = -1 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}-2}{2} \); when \( x = \pi \) we have \( f(\pi) = 1 \), and lastly, for \( x = \frac{7\pi}{4} \) we get \( f \left( \frac{7\pi}{4} \right) = -1 - \frac{\sqrt{2}}{2} = \frac{-2-\sqrt{2}}{2} \). We see \( f(x) > 0 \) on \( \left( \frac{\pi}{6}, \frac{\pi}{2} \right) \cup \left( \frac{5\pi}{6}, \frac{3\pi}{2} \right) \), so this is our answer. We can use the calculator to check that the graph of \( y = \sin(2x) \) is indeed above the graph of \( y = \cos(x) \) on those intervals.

\[
\begin{array}{cccccccc}
- & 0 & + & 0 & - & 0 & + & 0 & -
\end{array}
\]

\[
\begin{array}{cccccccc}
0 & \frac{\pi}{6} & \frac{\pi}{2} & \frac{5\pi}{6} & \frac{3\pi}{2} & 2\pi
\end{array}
\]

\( y = \sin(2x) \) and \( y = \cos(x) \)

3. Proceeding as in the last two problems, we rewrite \( \tan(x) \geq 3 \) as \( \tan(x) - 3 \geq 0 \) and let \( f(x) = \tan(x) - 3 \). We note that on \([0, 2\pi)\), \( f \) is undefined at \( x = \frac{\pi}{2} \) and \( \frac{3\pi}{2} \), so those values will need the usual disclaimer on the sign diagram. Moving along to zeros, solving \( f(x) = \tan(x) - 3 = 0 \) requires the arctangent function. We find \( x = \arctan(3) + \pi k \) for integers \( k \) and of these, only \( x = \arctan(3) \) and \( x = \arctan(3) + \pi \) lie in \([0, 2\pi)\). Since \( 3 > 0 \), we know \( 0 < \arctan(3) < \frac{\pi}{2} \) which allows us to position these zeros correctly on the sign diagram. To choose test values, we begin with \( x = 0 \) and find \( f(0) = -3 \). Finding a convenient test value in the interval \( (\arctan(3), \frac{\pi}{2}) \) is a bit more challenging. Keep in mind that the arctangent function is increasing and is bounded above by \( \frac{\pi}{2} \). This means that the number \( x = \arctan(117) \) is guaranteed to lie between \( \arctan(3) \) and \( \frac{\pi}{2} \). We see that \( f(\arctan(117)) = \tan(\arctan(117)) - 3 = 114 \). For our next test value, we take \( x = \pi \) and find \( f(\pi) = -3 \). To find our next test value, we note that since \( \arctan(3) < \arctan(117) < \frac{\pi}{2} \), it follows that \( \arctan(3) + \pi < \arctan(117) + \pi < \frac{3\pi}{2} \). Evaluating \( f \) at \( x = \arctan(117) + \pi \) yields \( f(\arctan(117) + \pi) = \tan(\arctan(117) + \pi) - 3 = \tan(\arctan(117)) - 3 = 114 \). We choose our last test value to be \( x = \frac{7\pi}{4} \) and find \( f \left( \frac{7\pi}{4} \right) = -4 \). Since we want \( f(x) \geq 0 \), we see that our answer is \( [\arctan(3), \frac{\pi}{2}) \cup [\arctan(3) + \pi, \frac{3\pi}{2}) \). Using the graphs of \( y = \tan(x) \) and \( y = 3 \), we see when the graph of the former is above (or meets) the graph of the latter.

---

11. See page 321 for a discussion of the non-standard character known as the interrobang.
12. We could have chosen any value \( \arctan(t) \) where \( t > 3 \).
13. . . . by adding \( \pi \) through the inequality . . .
Our next example puts solving equations and inequalities to good use – finding domains of functions.

**Example 10.7.4.** Express the domain of the following functions using extended interval notation.

1. \( f(x) = \csc \left( 2x + \frac{\pi}{3} \right) \)
2. \( f(x) = \frac{\sin(x)}{2\cos(x) - 1} \)
3. \( f(x) = \sqrt{1 - \cot(x)} \)

**Solution.**

1. To find the domain of \( f(x) = \csc \left( 2x + \frac{\pi}{3} \right) \), we rewrite \( f \) in terms of sine as \( f(x) = \frac{1}{\sin \left( 2x + \frac{\pi}{3} \right)} \).

   Since the sine function is defined everywhere, our only concern comes from zeros in the denominator. Solving \( \sin \left( 2x + \frac{\pi}{3} \right) = 0 \), we get \( x = -\frac{\pi}{6} + \frac{\pi}{2} k \) for integers \( k \). In set-builder notation, our domain is \( \left\{ x : x \neq -\frac{\pi}{6}, \frac{2\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{8\pi}{6}, ... \right\} \), where we have kept the denominators 6 throughout to help see the pattern. Graphing the situation on a numberline, we have

   ![Numberline](image)

   Proceeding as we did on page 756 in Section 10.3.1, we let \( x_k \) denote the \( k \)th number excluded from the domain and we have \( x_k = -\frac{\pi}{6} + \frac{\pi}{2} k = \frac{(3k-1)\pi}{6} \) for integers \( k \). The intervals which comprise the domain are of the form \( (x_k, x_{k+1}) = \left( \frac{(3k-1)\pi}{6}, \frac{(3k+2)\pi}{6} \right) \) as \( k \) runs through the integers. Using extended interval notation, we have that the domain is

   \[
   \bigcup_{k=-\infty}^{\infty} \left( \frac{(3k-1)\pi}{6}, \frac{(3k+2)\pi}{6} \right)
   \]

   We can check our answer by substituting in values of \( k \) to see that it matches our diagram.

---

14See page 756 for details about this notation.
2. Since the domains of $\sin(x)$ and $\cos(x)$ are all real numbers, the only concern when finding the domain of $f(x) = \frac{\sin(x)}{2 \cos(x) - 1}$ is division by zero so we set the denominator equal to zero and solve. From $2 \cos(x) - 1 = 0$ we get $\cos(x) = \frac{1}{2}$, so that $x = \frac{\pi}{3} + 2\pi k$ or $x = \frac{5\pi}{3} + 2\pi k$ for integers $k$. Using set-builder notation, the domain is $\{x : x \neq \frac{\pi}{3} + 2\pi k \text{ and } x \neq \frac{5\pi}{3} + 2\pi k \text{ for integers } k\}$, or $\{x : x \neq \pm \frac{\pi}{3}, \pm \frac{5\pi}{3}, \pm \frac{7\pi}{3}, \pm \frac{11\pi}{3}, \ldots\}$, so we have

![Graph showing the domain of the function](image)

Unlike the previous example, we have two different families of points to consider, and we present two ways of dealing with this kind of situation. One way is to generalize what we did in the previous example and use the formulas we found in our domain work to describe the intervals. To that end, we let $a_k = \frac{\pi}{3} + 2\pi k$, $b_k = \frac{5\pi}{3} + 2\pi k$ for integers $k$. The goal now is to write the domain in terms of the $a_k$'s and $b_k$'s. We find $a_0 = \frac{\pi}{3}$, $a_1 = \frac{7\pi}{3}$, $b_1 = \frac{11\pi}{3}$, $b_0 = \frac{5\pi}{3}$, $a_{-1} = \frac{13\pi}{3}$, $a_{-2} = \frac{11\pi}{3}$, $b_{-1} = -\frac{\pi}{3}$, $b_{-2} = -\frac{17\pi}{3}$. Hence, in terms of the $a_k$'s and $b_k$'s, our domain is

$$\ldots (a_{-2}, b_{-2}) \cup (b_{-2}, a_{-1}) \cup (a_{-1}, b_{-1}) \cup (b_{-1}, a_0) \cup (a_0, b_0) \cup (b_0, a_1) \cup (a_1, b_1) \cup \ldots$$

If we group these intervals in pairs, $(a_{-2}, b_{-2}) \cup (b_{-2}, a_{-1})$, $(a_{-1}, b_{-1}) \cup (b_{-1}, a_0)$, $(a_0, b_0) \cup (b_0, a_1)$ and so forth, we see a pattern emerge of the form $(a_k, b_k) \cup (b_k, a_{k+1})$ for integers $k$ so that our domain can be written as

$$\bigcup_{k=\infty}^{k=\infty} (a_k, b_k) \cup (b_k, a_{k+1}) = \bigcup_{k=\infty}^{k=\infty} \left(\frac{(6k+1)\pi}{3}, \frac{(6k+5)\pi}{3}\right) \cup \left(\frac{(6k+5)\pi}{3}, \frac{(6k+7)\pi}{3}\right)$$

A second approach to the problem exploits the periodic nature of $f$. Since $\cos(x)$ and $\sin(x)$ have period $2\pi$, it’s not too difficult to show the function $f$ repeats itself every $2\pi$ units. This means if we can find a formula for the domain on an interval of length $2\pi$, we can express the entire domain by translating our answer left and right on the $x$-axis by adding integer multiples of $2\pi$. One such interval that arises from our domain work is $[\frac{\pi}{3}, \frac{7\pi}{3}]$. The portion of the domain here is $[\frac{\pi}{3}, \frac{5\pi}{3}]$. Adding integer multiples of $2\pi$, we get the family of intervals $[\frac{\pi}{3} + 2\pi k, \frac{5\pi}{3} + 2\pi k] \cup [\frac{5\pi}{3} + 2\pi k, \frac{7\pi}{3} + 2\pi k]$ for integers $k$. We leave it to the reader to show that getting common denominators leads to our previous answer.

\textsuperscript{15}This doesn’t necessarily mean the period of $f$ is $2\pi$. The tangent function is comprised of $\cos(x)$ and $\sin(x)$, but its period is half theirs. The reader is invited to investigate the period of $f$. 

3. To find the domain of \( f(x) = \sqrt{1 - \cot(x)} \), we first note that, due to the presence of the \( \cot(x) \) term, \( x \neq \pi k \) for integers \( k \). Next, we recall that for the square root to be defined, we need \( 1 - \cot(x) \geq 0 \). Unlike the inequalities we solved in Example 10.7.3, we are not restricted here to a given interval. Our strategy is to solve this inequality over \((0, \pi)\) (the same interval which generates a fundamental cycle of cotangent) and then add integer multiples of the period, in this case, \( \pi \). We let \( g(x) = 1 - \cot(x) \) and set about making a sign diagram for \( g \) over the interval \((0, \pi)\) to find where \( g(x) \geq 0 \). We note that \( g \) is undefined for \( x = \pi k \) for integers \( k \), in particular, at the endpoints of our interval \( x = 0 \) and \( x = \pi \). Next, we look for the zeros of \( g \). Solving \( g(x) = 0 \), we get \( \cot(x) = 1 \) or \( x = \frac{\pi}{4} + \pi k \) for integers \( k \) and only one of these, \( x = \frac{\pi}{4} \), lies in \((0, \pi)\). Choosing the test values \( x = \frac{\pi}{6} \) and \( x = \frac{\pi}{2} \), we get \( g \left( \frac{\pi}{6} \right) = 1 - \sqrt{3} \), and \( g \left( \frac{\pi}{2} \right) = 1 \).

\[
\begin{array}{c|c|c|c}
\text{t} & (-) & 0 & (+) \\
0 & \frac{\pi}{4} & \pi
\end{array}
\]

We find \( g(x) \geq 0 \) on \([\frac{\pi}{4}, \pi)\). Adding multiples of the period we get our solution to consist of the intervals \( \left[ \frac{\pi}{4} + \pi k, \pi + \pi k \right] = \left[ \frac{(4k+1)\pi}{4}, (k+1)\pi \right) \). Using extended interval notation, we express our final answer as

\[
\bigcup_{k=-\infty}^{\infty} \left[ \frac{(4k+1)\pi}{4}, (k+1)\pi \right).
\]

We close this section with an example which demonstrates how to solve equations and inequalities involving the inverse trigonometric functions.

**Example 10.7.5.** Solve the following equations and inequalities analytically. Check your answers using a graphing utility.

1. \( \arcsin(2x) = \frac{\pi}{3} \)
2. \( 4 \arccos(x) - 3\pi = 0 \)
3. \( 3 \arccsc(2x - 1) + \pi = 2\pi \)
4. \( 4 \arctan^2(x) - 3\pi \arctan(x) - \pi^2 = 0 \)
5. \( \pi^2 - 4 \arccos^2(x) < 0 \)
6. \( 4 \arccot(3x) > \pi \)

**Solution.**

1. To solve \( \arcsin(2x) = \frac{\pi}{3} \), we first note that \( \frac{\pi}{3} \) is in the range of the arcsine function (so a solution exists!) Next, we exploit the inverse property of sine and arcsine from Theorem 10.26...
arcsin(2x) = \frac{\pi}{3} \\
\sin (\text{arcsin}(2x)) = \sin \left( \frac{\pi}{3} \right) \\
2x = \frac{\sqrt{3}}{2} \quad \text{Since } \sin(\text{arcsin}(u)) = u \\
x = \frac{\sqrt{3}}{4}

Graphing \( y = \text{arcsin}(2x) \) and the horizontal line \( y = \frac{\pi}{3} \), we see they intersect at \( \frac{\sqrt{3}}{4} \approx 0.4430 \).

2. Our first step in solving \( 4 \cos^{-1}(x) - 3\pi = 0 \) is to isolate the arccosine. Doing so, we get \( \cos^{-1}(x) = \frac{3\pi}{4} \). Since \( \frac{3\pi}{4} \) is in the range of arccosine, we may apply Theorem 10.26

\[
\begin{align*}
\cos^{-1}(x) &= \frac{3\pi}{4} \\
\cos \left( \cos^{-1}(x) \right) &= \cos \left( \frac{3\pi}{4} \right) \\
x &= -\frac{\sqrt{2}}{2} \quad \text{Since } \cos(\cos^{-1}(u)) = u
\end{align*}
\]

The calculator confirms \( y = 4 \cos^{-1}(x) - 3\pi \) crosses \( y = 0 \) (the \( x \)-axis) at \( -\frac{\sqrt{2}}{2} \approx -0.7071 \).

3. From \( 3 \sec^{-1}(2x - 1) + \pi = 2\pi \), we get \( \sec^{-1}(2x - 1) = \frac{\pi}{3} \). As we saw in Section 10.6, there are two possible ranges for the arccosecant function. Fortunately, both ranges contain \( \frac{\pi}{3} \).

Applying Theorem 10.28 / 10.29, we get

\[
\begin{align*}
\sec^{-1}(2x - 1) &= \frac{\pi}{3} \\
\sec \left( \sec^{-1}(2x - 1) \right) &= \sec \left( \frac{\pi}{3} \right) \\
2x - 1 &= 2 \quad \text{Since } \sec(\sec^{-1}(u)) = u \\
x &= \frac{3}{2}
\end{align*}
\]

To check using our calculator, we need to graph \( y = 3 \sec^{-1}(2x - 1) + \pi \). To do so, we make use of the identity \( \sec^{-1}(u) = \arccos \left( \frac{1}{u} \right) \) from Theorems 10.28 and 10.29.\(^{16}\) We see the graph of \( y = 3 \arccos \left( \frac{1}{2x-1} \right) + \pi \) and the horizontal line \( y = 2\pi \) intersect at \( \frac{3}{2} = 1.5 \).

\(^{16}\)Since we are checking for solutions where arccosecant is positive, we know \( u = 2x - 1 \geq 1 \), and so the identity applies in both cases.
4. With the presence of both \( \arctan^2(x) \) and \( \arctan(x) \), we substitute \( u = \arctan(x) \). The equation \( 4 \arctan^2(x) - 3\pi \arctan(x) - \pi^2 = 0 \) becomes \( 4u^2 - 3\pi u - \pi^2 = 0 \). Factoring, we get \((4u + \pi)(u - \pi) = 0\), so \( u = \arctan(x) = -\frac{\pi}{4} \) or \( u = \arctan(x) = \pi \). Since \(-\frac{\pi}{4}\) is in the range of arctangent, but \(\pi\) is not, we only get solutions from the first equation. Using Theorem 10.27, we get:

\[
\begin{align*}
\arctan(x) & = -\frac{\pi}{4} \\
\tan(\arctan(x)) & = \tan\left(-\frac{\pi}{4}\right) \\
x & = -1 \quad \text{Since } \tan(\arctan(u)) = u.
\end{align*}
\]

The calculator verifies our result.

5. Since the inverse trigonometric functions are continuous on their domains, we can solve inequalities featuring these functions using sign diagrams. Since all of the nonzero terms of \( \pi^2 - 4 \arccos^2(x) < 0 \) are on one side of the inequality, we let \( f(x) = \pi^2 - 4 \arccos^2(x) \) and note the domain of \( f \) is limited by the \( \arccos(x) \) to \([-1, 1]\). Next, we find the zeros of \( f \) by setting \( f(x) = \pi^2 - 4 \arccos^2(x) = 0 \). We get \( \arccos(x) = \pm\frac{\pi}{2} \), and since the range of \( \arccos(x) \) is \([0, \pi]\), we focus our attention on \( \arccos(x) = \frac{\pi}{2} \). Using Theorem 10.26, we get \( x = \cos\left(\frac{\pi}{2}\right) = 0 \) as our only zero. Hence, we have two test intervals, \([-1, 0) \) and \((0, 1]\). Choosing test values \( x = \pm1 \), we get \( f(-1) = -3\pi^2 < 0 \) and \( f(1) = \pi^2 > 0 \). Since we are looking for where \( f(x) = \pi^2 - 4 \arccos^2(x) < 0 \), our answer is \([-1, 0) \). The calculator confirms that for these values of \( x \), the graph of \( y = \pi^2 - 4 \arccos^2(x) \) is below \( y = 0 \) (the \( x \)-axis.)

\[\text{It’s not as bad as it looks... don’t let the } \pi \text{ throw you!} \]
6. To begin, we rewrite $4 \arccot(3x) > \pi$ as $4 \arccot(3x) - \pi > 0$. We let $f(x) = 4 \arccot(3x) - \pi$, and note the domain of $f$ is all real numbers, $(-\infty, \infty)$. To find the zeros of $f$, we set $f(x) = 4 \arccot(3x) - \pi = 0$ and solve. We get $\arccot(3x) = \frac{\pi}{4}$, and since $\frac{\pi}{4}$ is in the range of arccotangent, we may apply Theorem 10.27 and solve

$\arccot(3x) = \frac{\pi}{4}$\]
\[\cot(\arccot(3x)) = \cot \left( \frac{\pi}{4} \right)\]
\[3x = 1 \quad \text{Since } \cot(\arccot(u)) = u.\]
\[x = \frac{1}{3}\]

Next, we make a sign diagram for $f$. Since the domain of $f$ is all real numbers, and there is only one zero of $f$, $x = \frac{1}{3}$, we have two test intervals, $(-\infty, \frac{1}{3})$ and $(\frac{1}{3}, \infty)$. Ideally, we wish to find test values $x$ in these intervals so that $\arccot(4x)$ corresponds to one of our oft-used 'common' angles. After a bit of computation, we choose $x = 0$ for $x < \frac{1}{3}$ and for $x > \frac{1}{3}$, we choose $x = \frac{\sqrt{3}}{3}$. We find $f(0) = \pi > 0$ and $f \left( \frac{\sqrt{3}}{3} \right) = -\frac{\pi}{3} < 0$. Since we are looking for where $f(x) = 4 \arccot(3x) - \pi > 0$, we get our answer $(-\infty, \frac{1}{3})$. To check graphically, we use the technique in number 2c of Example 10.6.5 in Section 10.6 to graph $y = 4 \arccot(3x)$ and we see it is above the horizontal line $y = \pi$ on $(-\infty, \frac{1}{3}) = (-\infty, 0.3)$.

---

\[\frac{\sqrt{3}}{3} \]

\[(+) 0 (-)\]

\[y = 4 \arccot(3x) \quad \text{and } y = \pi\]

\[\text{Set } 3x \text{ equal to the cotangents of the 'common angles' and choose accordingly.}\]
Chapter 11

Applications of Trigonometry

11.1 Applications of Sinusoids

In the same way exponential functions can be used to model a wide variety of phenomena in nature, the cosine and sine functions can be used to model their fair share of natural behaviors. In section 10.5, we introduced the concept of a sinusoid as a function which can be written either in the form $C(x) = A \cos(\omega x + \phi) + B$ for $\omega > 0$ or equivalently, in the form $S(x) = A \sin(\omega x + \phi) + B$ for $\omega > 0$. At the time, we remained undecided as to which form we preferred, but the time for such indecision is over. For clarity of exposition we focus on the sine function in this section and switch to the independent variable $t$, since the applications in this section are time-dependent. We reintroduce and summarize all of the important facts and definitions about this form of the sinusoid below.

<table>
<thead>
<tr>
<th>Properties of the Sinusoid $S(t) = A \sin(\omega t + \phi) + B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>• The amplitude is $</td>
</tr>
<tr>
<td>• The angular frequency is $\omega$ and the ordinary frequency is $f = \frac{\omega}{2\pi}$</td>
</tr>
<tr>
<td>• The period is $T = \frac{1}{f} = \frac{2\pi}{\omega}$</td>
</tr>
<tr>
<td>• The phase is $\phi$ and the phase shift is $-\frac{\phi}{\omega}$</td>
</tr>
<tr>
<td>• The vertical shift or baseline is $B$</td>
</tr>
</tbody>
</table>

Along with knowing these formulas, it is helpful to remember what these quantities mean in context. The amplitude measures the maximum displacement of the sine wave from its baseline (determined by the vertical shift), the period is the length of time it takes to complete one cycle of the sinusoid, the angular frequency tells how many cycles are completed over an interval of length $2\pi$, and the ordinary frequency measures how many cycles occur per unit of time. The phase indicates what

---

1 See Section 6.5.
2 Sine haters can use the co-function identity $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$ to turn all of the sines into cosines.
angle $\phi$ corresponds to $t = 0$, and the phase shift represents how much of a ‘head start’ the sinusoid has over the un-shifted sine function. The figure below is repeated from Section 10.5.

In Section 10.1.1, we introduced the concept of circular motion and in Section 10.2.1, we developed formulas for circular motion. Our first foray into sinusoidal motion puts these notions to good use.

Example 11.1.1. Recall from Exercise 55 in Section 10.1 that The Giant Wheel at Cedar Point is a circle with diameter 128 feet which sits on an 8 foot tall platform making its overall height 136 feet. It completes two revolutions in 2 minutes and 7 seconds. Assuming that the riders are at the edge of the circle, find a sinusoid which describes the height of the passengers above the ground $t$ seconds after they pass the point on the wheel closest to the ground.

Solution. We sketch the problem situation below and assume a counter-clockwise rotation.$^3$

---

$^3$Otherwise, we could just observe the motion of the wheel from the other side.
We know from the equations given on page 732 in Section 10.2.1 that the $y$-coordinate for counterclockwise motion on a circle of radius $r$ centered at the origin with constant angular velocity (frequency) $\omega$ is given by $y = r \sin(\omega t)$. Here, $t = 0$ corresponds to the point $(r, 0)$ so that $\theta$, the angle measuring the amount of rotation, is in standard position. In our case, the diameter of the wheel is 128 feet, so the radius is $r = 64$ feet. Since the wheel completes two revolutions in 2 minutes and 7 seconds (which is 127 seconds) the period $T = \frac{1}{2}(127) = \frac{127}{2}$ seconds. Hence, the angular frequency is $\omega = \frac{2\pi}{T} = \frac{4\pi}{127}$ radians per second. Putting these two pieces of information together, we have that $y = 64 \sin \left( \frac{4\pi}{127} t \right)$ describes the $y$-coordinate on the Giant Wheel after $t$ seconds, assuming it is centered at $(0, 0)$ with $t = 0$ corresponding to the point $Q$. In order to find an expression for $h$, we take the point $O$ in the figure as the origin. Since the base of the Giant Wheel ride is 8 feet above the ground and the Giant Wheel itself has a radius of 64 feet, its center is 72 feet above the ground. To account for this vertical shift upward,$^4$ we add 72 to our formula for $y$ to obtain the new formula $h = y + 72 = 64 \sin \left( \frac{4\pi}{127} t \right) + 72$. Next, we need to adjust things so that $t = 0$ corresponds to the point $P$ instead of the point $Q$. This is where the phase comes into play. Geometrically, we need to shift the angle $\theta$ in the figure back $\frac{\pi}{2}$ radians. From Section 10.2.1, we know $\theta = \omega t = \frac{4\pi}{127} t$, so we (temporarily) write the height in terms of $\theta$ as $h = 64 \sin (\theta) + 72$. Subtracting $\frac{\pi}{2}$ from $\theta$ gives the final answer $h(t) = 64 \sin \left( \theta - \frac{\pi}{2} \right) + 72 = 64 \sin \left( \frac{4\pi}{127} t - \frac{\pi}{2} \right) + 72$. We can check the reasonableness of our answer by graphing $y = h(t)$ over the interval $[0, \frac{127}{2}]$.

A few remarks about Example 11.1.1 are in order. First, note that the amplitude of 64 in our answer corresponds to the radius of the Giant Wheel. This means that passengers on the Giant Wheel never stray more than 64 feet vertically from the center of the Wheel, which makes sense. Second, the phase shift of our answer works out to be $\frac{\pi/2}{4\pi/127} = \frac{127}{8} = 15.875$. This represents the ‘time delay’ (in seconds) we introduce by starting the motion at the point $P$ as opposed to the point $Q$. Said differently, passengers which ‘start’ at $P$ take 15.875 seconds to ‘catch up’ to the point $Q$.

Our next example revisits the daylight data first introduced in Section 2.5, Exercise 6b.

$^4$We are readjusting our ‘baseline’ from $y = 0$ to $y = 72$.
Example 11.1.2. According to the U.S. Naval Observatory website, the number of hours $H$ of daylight that Fairbanks, Alaska received on the 21st day of the $n$th month of 2009 is given below. Here $t = 1$ represents January 21, 2009, $t = 2$ represents February 21, 2009, and so on.

<table>
<thead>
<tr>
<th>Month Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hours of Daylight</td>
<td>5.8</td>
<td>9.3</td>
<td>12.4</td>
<td>15.9</td>
<td>19.4</td>
<td>21.8</td>
<td>19.4</td>
<td>15.6</td>
<td>12.4</td>
<td>9.1</td>
<td>5.6</td>
<td>3.3</td>
</tr>
</tbody>
</table>

1. Find a sinusoid which models these data and use a graphing utility to graph your answer along with the data.

2. Compare your answer to part 1 to one obtained using the regression feature of a calculator.

Solution.

1. To get a feel for the data, we plot it below.

The data certainly appear sinusoidal, but when it comes down to it, fitting a sinusoid to data manually is not an exact science. We do our best to find the constants $A$, $\omega$, $\phi$ and $B$ so that the function $H(t) = A \sin(\omega t + \phi) + B$ closely matches the data. We first go after the vertical shift $B$ whose value determines the baseline. In a typical sinusoid, the value of $B$ is the average of the maximum and minimum values. So here we take $B = \frac{3.3 + 21.8}{2} = 12.55$. Next is the amplitude $A$ which is the displacement from the baseline to the maximum (and minimum) values. We find $A = 21.8 - 12.55 = 12.55 - 3.3 = 9.25$. At this point, we have $H(t) = 9.25 \sin(\omega t + \phi) + 12.55$. Next, we go after the angular frequency $\omega$. Since the data collected is over the span of a year (12 months), we take the period $T = 12$ months.

---

5Okay, it appears to be the ‘∧’ shape we saw in some of the graphs in Section 2.2. Just humor us.

6Even though the data collected lies in the interval $[1, 12]$, which has a length of 11, we need to think of the data point at $t = 1$ as a representative sample of the amount of daylight for every day in January. That is, it represents $H(t)$ over the interval $[0, 1]$. Similarly, $t = 2$ is a sample of $H(t)$ over $[1, 2]$, and so forth.
11.1 Applications of Sinusoids

means $\omega = \frac{2\pi}{T} = \frac{2\pi}{12} = \frac{\pi}{6}$. The last quantity to find is the phase $\phi$. Unlike the previous example, it is easier in this case to find the phase shift $-\phi \omega$. Since we picked $A > 0$, the phase shift corresponds to the first value of $t$ with $H(t) = 12.55$ (the baseline value).\(^7\) Here, we choose $t = 3$, since its corresponding $H$ value of 12.4 is closer to 12.55 than the next value, 15.9, which corresponds to $t = 4$. Hence, $-\frac{\phi}{\omega} = 3$, so $\phi = -3\omega = -3\left(\frac{\pi}{6}\right) = -\frac{\pi}{2}$. We have $H(t) = 9.25 \sin \left(\frac{\pi}{6} t - \frac{\pi}{2}\right) + 12.55$. Below is a graph of our data with the curve $y = H(t)$.

2. Using the ‘SinReg’ command, we graph the calculator’s regression below.

While both models seem to be reasonable fits to the data, the calculator model is possibly the better fit. The calculator does not give us an $r^2$ value like it did for linear regressions in Section 2.5, nor does it give us an $R^2$ value like it did for quadratic, cubic and quartic regressions as in Section 3.1. The reason for this, much like the reason for the absence of $R^2$ for the logistic model in Section 6.5, is beyond the scope of this course. We’ll just have to use our own good judgment when choosing the best sinusoid model.

11.1.1 Harmonic Motion

One of the major applications of sinusoids in Science and Engineering is the study of harmonic motion. The equations for harmonic motion can be used to describe a wide range of phenomena, from the motion of an object on a spring, to the response of an electronic circuit. In this subsection, we restrict our attention to modeling a simple spring system. Before we jump into the Mathematics, there are some Physics terms and concepts we need to discuss. In Physics, ‘mass’ is defined as a measure of an object’s resistance to straight-line motion whereas ‘weight’ is the amount of force (pull) gravity exerts on an object. An object’s mass cannot change,\(^8\) while its weight could change.

\(^7\)See the figure on page 882.

\(^8\)Well, assuming the object isn’t subjected to relativistic speeds . . .
An object which weighs 6 pounds on the surface of the Earth would weigh 1 pound on the surface of the Moon, but its mass is the same in both places. In the English system of units, ‘pounds’ (lbs.) is a measure of force (weight), and the corresponding unit of mass is the ‘slug’. In the SI system, the unit of force is ‘Newtons’ (N) and the associated unit of mass is the ‘kilogram’ (kg). We convert between mass and weight using the formula \( w = mg \). Here, \( w \) is the weight of the object, \( m \) is the mass and \( g \) is the acceleration due to gravity. In the English system, \( g = 32 \text{ feet/second}^2 \), and in the SI system, \( g = 9.8 \text{ meters/second}^2 \). Hence, on Earth a mass of 1 slug weighs 32 lbs. and a mass of 1 kg weighs 9.8 N. Suppose we attach an object with mass \( m \) to a spring as depicted below. The weight of the object will stretch the spring. The system is said to be in ‘equilibrium’ when the weight of the object is perfectly balanced with the restorative force of the spring. How far the spring stretches to reach equilibrium depends on the spring’s ‘spring constant’. Usually denoted by the letter \( k \), the spring constant relates the force \( F \) applied to the spring to the amount \( d \) the spring stretches in accordance with Hooke’s Law \( F = kd \). If the object is released above or below the equilibrium position, or if the object is released with an upward or downward velocity, the object will bounce up and down on the end of the spring until some external force stops it. If we let \( x(t) \) denote the object’s displacement from the equilibrium position at time \( t \), then \( x(t) = 0 \) means the object is at the equilibrium position, \( x(t) < 0 \) means the object is above the equilibrium position, and \( x(t) > 0 \) means the object is below the equilibrium position. The function \( x(t) \) is called the ‘equation of motion’ of the object.

\[ x(t) = 0 \text{ at the equilibrium position} \]
\[ x(t) < 0 \text{ above the equilibrium position} \]
\[ x(t) > 0 \text{ below the equilibrium position} \]

If we ignore all other influences on the system except gravity and the spring force, then Physics tells us that gravity and the spring force will battle each other forever and the object will oscillate indefinitely. In this case, we describe the motion as ‘free’ (meaning there is no external force causing the motion) and ‘undamped’ (meaning we ignore friction caused by surrounding medium, which in our case is air). The following theorem, which comes from Differential Equations, gives \( x(t) \) as a function of the mass \( m \) of the object, the spring constant \( k \), the initial displacement \( x_0 \) of the object.

---

\(^9\)This is a consequence of Newton’s Second Law of Motion \( F = ma \) where \( F \) is force, \( m \) is mass and \( a \) is acceleration.

\(^{10}\)Note that 1 pound = 1 \( \text{slug foot/second}^2 \) and 1 Newton = 1 \( \text{kg meter/second}^2 \).

\(^{11}\)Look familiar? We saw Hooke’s Law in Section 4.3.1.

\(^{12}\)To keep units compatible, if we are using the English system, we use feet (ft.) to measure displacement. If we are in the SI system, we measure displacement in meters (m). Time is always measured in seconds (s).
11.1 Applications of Sinusoids

object and initial velocity \( v_0 \) of the object. As with \( x(t), x_0 = 0 \) means the object is released from the equilibrium position, \( x_0 < 0 \) means the object is released above the equilibrium position and \( x_0 > 0 \) means the object is released below the equilibrium position. As far as the initial velocity \( v_0 \) is concerned, \( v_0 = 0 \) means the object is released from rest, \( v_0 < 0 \) means the object is heading upwards and \( v_0 > 0 \) means the object is heading downwards.\(^\text{13}\)

\[ \text{Theorem 11.1. Equation for Free Undamped Harmonic Motion:} \]

Suppose an object of mass \( m \) is suspended from a spring with spring constant \( k \). If the initial displacement from the equilibrium position is \( x_0 \) and the initial velocity of the object is \( v_0 \), then the displacement \( x \) from the equilibrium position at time \( t \) is given by \( x(t) = A\sin(\omega t + \phi) \) where

- \( \omega = \sqrt{\frac{k}{m}} \) and \( A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} \)
- \( A\sin(\phi) = x_0 \) and \( A\omega\cos(\phi) = v_0 \).

It is a great exercise in ‘dimensional analysis’ to verify that the formulas given in Theorem 11.1 work out so that \( \omega \) has units \( \frac{1}{s} \) and \( A \) has units ft. or m, depending on which system we choose.

\[ \text{Example 11.1.3. Suppose an object weighing 64 pounds stretches a spring 8 feet.} \]

1. If the object is attached to the spring and released 3 feet below the equilibrium position from rest, find the equation of motion of the object, \( x(t) \). When does the object first pass through the equilibrium position? Is the object heading upwards or downwards at this instant?

2. If the object is attached to the spring and released 3 feet below the equilibrium position with an upward velocity of 8 feet per second, find the equation of motion of the object, \( x(t) \). What is the longest distance the object travels above the equilibrium position? When does this first happen? Confirm your result using a graphing utility.

\[ \text{Solution.} \] In order to use the formulas in Theorem 11.1, we first need to determine the spring constant \( k \) and the mass of the object \( m \). To find \( k \), we use Hooke’s Law \( F = kd \). We know the object weighs 64 lbs. and stretches the spring 8 ft. Using \( F = 64 \) and \( d = 8 \), we get \( 64 = k \cdot 8 \), or \( k = 8 \text{ lb}\cdot\text{ft.} \). To find \( m \), we use \( w = mg \) with \( w = 64 \text{ lbs.} \) and \( g = 32 \text{ ft.}^2/s^2 \). We get \( m = 2 \text{ slugs} \). We can now proceed to apply Theorem 11.1.

1. With \( k = 8 \) and \( m = 2 \), we get \( \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{2}} = 2 \). We are told that the object is released 3 feet below the equilibrium position ‘from rest.’ This means \( x_0 = 3 \) and \( v_0 = 0 \). Therefore, \( A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} = \sqrt{3^2 + 0^2} = 3 \). To determine the phase \( \phi \), we have \( A\sin(\phi) = x_0 \), which in this case gives \( 3\sin(\phi) = 3 \) so \( \sin(\phi) = 1 \). Only \( \phi = \frac{\pi}{2} \) and angles coterminal to it:

\[ \text{The sign conventions here are carried over from Physics. If not for the spring, the object would fall towards the ground, which is the ‘natural’ or ‘positive’ direction. Since the spring force acts in direct opposition to gravity, any movement upwards is considered ‘negative’.} \]
satisfy this condition, so we pick\(^\text{14}\) the phase to be \(\phi = \frac{\pi}{2}\). Hence, the equation of motion is \(x(t) = 3 \sin \left(2t + \frac{\pi}{2}\right)\). To find when the object passes through the equilibrium position we solve \(x(t) = 3 \sin \left(2t + \frac{\pi}{2}\right) = 0\). Going through the usual analysis we find \(t = -\frac{\pi}{4} + \frac{\pi}{2}k\) for integers \(k\). Since we are interested in the first time the object passes through the equilibrium position, we look for the smallest positive \(t\) value which in this case is \(t = \frac{\pi}{4} \approx 0.78\) seconds after the start of the motion. Common sense suggests that if we release the object below the equilibrium position, the object should be traveling upwards when it first passes through it. To check this answer, we graph one cycle of \(x(t)\). Since our applied domain in this situation is \(t \geq 0\), and the period of \(x(t)\) is \(T = \frac{2\pi}{\omega} = \frac{\pi}{2\pi} = \pi\), we graph \(x(t)\) over the interval \([0, \pi]\). Remembering that \(x(t) > 0\) means the object is below the equilibrium position and \(x(t) < 0\) means the object is above the equilibrium position, the fact our graph is crossing through the \(t\)-axis from positive \(x\) to negative \(x\) at \(t = \frac{\pi}{4}\) confirms our answer.

2. The only difference between this problem and the previous problem is that we now release the object with an upward velocity of \(\frac{8}{5}\) ft/s. We still have \(\omega = 2\) and \(x_0 = 3\), but now we have \(v_0 = -8\), the negative indicating the velocity is directed upwards. Here, we get \(A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} = \sqrt{3^2 + (-4)^2} = 5\). From \(A \sin(\phi) = x_0\), we get \(5 \sin(\phi) = 3\) which gives \(\sin(\phi) = \frac{3}{5}\). From \(A \omega \cos(\phi) = v_0\), we get \(10 \cos(\phi) = -8\), or \(\cos(\phi) = -\frac{4}{5}\). This means that \(\phi\) is a Quadrant II angle which we can describe in terms of either arcsine or arccosine. Since \(x(t)\) is expressed in terms of sine, we choose to express \(\phi = \pi - \arcsin\left(\frac{3}{5}\right)\). Hence, \(x(t) = 5 \sin \left(2t + \left[\pi - \arcsin\left(\frac{3}{5}\right)\right]\right)\). Since the amplitude of \(x(t)\) is 5, the object will travel at most 5 feet above the equilibrium position. To find when this happens, we solve the equation \(x(t) = 5 \sin \left(2t + \left[\pi - \arcsin\left(\frac{3}{5}\right)\right]\right) = -5\), the negative once again signifying that the object is above the equilibrium position. Going through the usual machinations, we get \(t = -\frac{1}{2} \arcsin\left(\frac{3}{5}\right) + \frac{\pi}{4} + \pi k\) for integers \(k\). The smallest of these values occurs when \(k = 0\), that is, \(t = -\frac{1}{2} \arcsin\left(\frac{3}{5}\right) + \frac{\pi}{4} \approx 1.107\) seconds after the start of the motion. To check our answer using the calculator, we graph \(y = 5 \sin \left(2x + \left[\pi - \arcsin\left(\frac{3}{5}\right)\right]\right)\) on a graphing utility and confirm the coordinates of the first relative minimum to be approximately \((1.107, -5)\).

\[ x(t) = 3 \sin \left(2t + \frac{\pi}{2}\right) \]

\[ y = 5 \sin \left(2x + \left[\pi - \arcsin\left(\frac{3}{5}\right)\right]\right) \]

\[ x = 1.1071484 y = -5 \]

It is possible, though beyond the scope of this course, to model the effects of friction and other external forces acting on the system.\(^\text{15}\) While we may not have the Physics and Calculus background

\(^{14}\)For confirmation, we note that \(A \omega \cos(\phi) = v_0\), which in this case reduces to \(6 \cos(\phi) = 0\).

\(^{15}\)Take a good Differential Equations class to see this!
11.1 Applications of Sinusoids

To derive equations of motion for these scenarios, we can certainly analyze them. We examine three cases in the following example.

**Example 11.1.4.**

1. Write \( x(t) = 5e^{-t/5} \cos(t) + 5e^{-t/5} \sqrt{3} \sin(t) \) in the form \( x(t) = A(t) \sin(\omega t + \phi) \). Graph \( x(t) \) using a graphing utility.

2. Write \( x(t) = (t + 3) \sqrt{2} \cos(2t) + (t + 3) \sqrt{2} \sin(2t) \) in the form \( x(t) = A(t) \sin(\omega t + \phi) \). Graph \( x(t) \) using a graphing utility.

3. Find the period of \( x(t) = 5 \sin(6t) - 5 \sin(8t) \). Graph \( x(t) \) using a graphing utility.

**Solution.**

1. We start rewriting \( x(t) = 5e^{-t/5} \cos(t) + 5e^{-t/5} \sqrt{3} \sin(t) \) by factoring out \( 5e^{-t/5} \) from both terms to get \( x(t) = 5e^{-t/5} (\cos(t) + \sqrt{3} \sin(t)) \). We convert what’s left in parentheses to the required form using the formulas introduced in Exercise 36 from Section 10.5. We find \( (\cos(t) + \sqrt{3} \sin(t)) = 2 \sin(t + \pi/3) \) so that \( x(t) = 10e^{-t/5} \sin(t + \pi/3) \). Graphing this on the calculator as \( y = 10e^{-x/5} \sin(x + \pi/3) \) reveals some interesting behavior. The sinusoidal nature continues indefinitely, but it is being attenuated. In the sinusoid \( A \sin(\omega x + \phi) \), the coefficient \( A \) of the sine function is the amplitude. In the case of \( y = 10e^{-x/5} \sin(x + \pi/3) \), we can think of the amplitude function here is \( A(x) = 2(x + 3) = 2x + 6 \), which continues to grow without bound.

2. Proceeding as in the first example, we factor out \( (t + 3) \sqrt{2} \) from each term in the function \( x(t) = (t + 3) \sqrt{2} \cos(2t) + (t + 3) \sqrt{2} \sin(2t) \) to get \( x(t) = (t + 3) \sqrt{2} (\cos(2t) + \sin(2t)) \). We find \( (\cos(2t) + \sin(2t)) = \sqrt{2} \sin(2t + \pi/4) \), so \( x(t) = 2(t + 3) \sin(2t + \pi/4) \). Graphing this on the calculator as \( y = 2(x + 3) \sin(2x + \pi/4) \), we find the sinusoid’s amplitude growing. Since our amplitude function here is \( A(x) = 2(x + 3) = 2x + 6 \), which continues to grow without bound.
as $x \to \infty$, this is hardly surprising. The phenomenon illustrated here is ‘forced’ motion. That is, we imagine that the entire apparatus on which the spring is attached is oscillating as well. In this case, we are witnessing a ‘resonance’ effect – the frequency of the external oscillation matches the frequency of the motion of the object on the spring.\(^{16}\)

\[ y = 2(x + 3) \sin \left( 2x + \frac{\pi}{4} \right) \]

\[ y = 2(x + 3) \sin \left( 2x + \frac{\pi}{4} \right) \]

\[ y = \pm 2(x + 3) \]

3. Last, but not least, we come to $x(t) = 5 \sin(6t) - 5 \sin(8t)$. To find the period of this function, we need to determine the length of the smallest interval on which both $f(t) = 5\sin(6t)$ and $g(t) = 5\sin(8t)$ complete a whole number of cycles. To do this, we take the ratio of their frequencies and reduce to lowest terms: $\frac{6}{8} = \frac{3}{4}$. This tells us that for every 3 cycles $f$ makes, $g$ makes 4. In other words, the period of $x(t)$ is three times the period of $f(t)$ (which is four times the period of $g(t)$), or $\pi$. We graph $y = 5\sin(6x) - 5\sin(8x)$ over $[0, \pi]$ on the calculator to check this. This equation of motion also results from ‘forced’ motion, but here the frequency of the external oscillation is different than that of the object on the spring. Since the sinusoids here have different frequencies, they are ‘out of sync’ and do not amplify each other as in the previous example. Taking things a step further, we can use a sum to product identity to rewrite $x(t) = 5 \sin(6t) - 5 \sin(8t)$ as $x(t) = -10 \sin(t) \cos(7t)$. The lower frequency factor in this expression, $-10 \sin(t)$, plays an interesting role in the graph of $x(t)$. Below we graph $y = 5\sin(6x) - 5\sin(8x)$ and $y = \pm 10 \sin(x)$ over $[0, 2\pi]$. This is an example of the ‘beat’ phenomena, and the curious reader is invited to explore this concept as well.\(^{17}\)

\[ y = 5 \sin(6x) - 5 \sin(8x) \text{ over } [0, \pi] \]

\[ y = 5 \sin(6x) - 5 \sin(8x) \text{ and } y = \pm 10 \sin(x) \text{ over } [0, 2\pi] \]

\(^{16}\)The reader is invited to investigate the destructive implications of resonance.

\(^{17}\)A good place to start is this article on beats.
11.2 The Law of Sines

Trigonometry literally means ‘measuring triangles’ and with Chapter 10 under our belts, we are more than prepared to do just that. The main goal of this section and the next is to develop theorems which allow us to ‘solve’ triangles – that is, find the length of each side of a triangle and the measure of each of its angles. In Sections 10.2, 10.3 and 10.6, we’ve had some experience solving right triangles. The following example reviews what we know.

Example 11.2.1. Given a right triangle with a hypotenuse of length 7 units and one leg of length 4 units, find the length of the remaining side and the measures of the remaining angles. Express the angles in decimal degrees, rounded to the nearest hundredth of a degree.

Solution. For definitiveness, we label the triangle below.

To find the length of the missing side \(a\), we use the Pythagorean Theorem to get

\[ a^2 + 4^2 = 7^2 \]

which then yields \(a = \sqrt{33}\) units. Now that all three sides of the triangle are known, there are several ways we can find \(\alpha\) using the inverse trigonometric functions. To decrease the chances of propagating error, however, we stick to using the data given to us in the problem. In this case, the lengths 4 and 7 were given, so we want to relate these to \(\alpha\). According to Theorem 10.4, \(\cos(\alpha) = \frac{4}{7}\).

Since \(\alpha\) is an acute angle, \(\alpha = \arccos\left(\frac{4}{7}\right)\) radians. Converting to degrees, we find \(\alpha \approx 55.15^\circ\). Now that we have the measure of angle \(\alpha\), we could find the measure of angle \(\beta\) using the fact that \(\alpha + \beta = 90^\circ\). Once again, we opt to use the data given to us in the problem. According to Theorem 10.4, we have that \(\sin(\beta) = \frac{4}{7}\) so \(\beta = \arcsin\left(\frac{4}{7}\right)\) radians and we have \(\beta \approx 34.85^\circ\).

A few remarks about Example 11.2.1 are in order. First, we adhere to the convention that a lower case Greek letter denotes an angle\(^1\) and the corresponding lowercase English letter represents the side\(^2\) opposite that angle. Thus, \(a\) is the side opposite \(\alpha\), \(b\) is the side opposite \(\beta\) and \(c\) is the side opposite \(\gamma\). Taken together, the pairs \((\alpha, a)\), \((\beta, b)\) and \((\gamma, c)\) are called angle-side opposite pairs. Second, as mentioned earlier, we will strive to solve for quantities using the original data given in the problem whenever possible. While this is not always the easiest or fastest way to proceed, it

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\(^1\)as well as the measure of said angle

\(^2\)as well as the length of said side
minimizes the chances of propagated error. Third, since many of the applications which require solving triangles ‘in the wild’ rely on degree measure, we shall adopt this convention for the time being. The Pythagorean Theorem along with Theorems 10.4 and 10.10 allow us to easily handle any given right triangle problem, but what if the triangle isn’t a right triangle? In certain cases, we can use the Law of Sines to help.

**Theorem 11.2. The Law of Sines:** Given a triangle with angle-side opposite pairs \((\alpha, a)\), \((\beta, b)\) and \((\gamma, c)\), the following ratios hold

\[
\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}
\]

or, equivalently,

\[
\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}
\]

The proof of the Law of Sines can be broken into three cases. For our first case, consider the triangle \(\triangle ABC\) below, all of whose angles are acute, with angle-side opposite pairs \((\alpha, a)\), \((\beta, b)\) and \((\gamma, c)\). If we drop an altitude from vertex \(B\), we divide the triangle into two right triangles: \(\triangle ABQ\) and \(\triangle BCQ\). If we call the length of the altitude \(h\) (for height), we get from Theorem 10.4 that \(\sin(\alpha) = \frac{h}{c}\) and \(\sin(\gamma) = \frac{h}{a}\) so that \(h = c \sin(\alpha) = a \sin(\gamma)\). After some rearrangement of the last equation, we get \(\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}\). If we drop an altitude from vertex \(A\), we can proceed as above using the triangles \(\triangle ABQ\) and \(\triangle ACQ\) to get \(\frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}\), completing the proof for this case.

For our next case consider the triangle \(\triangle ABC\) below with obtuse angle \(\alpha\). Extending an altitude from vertex \(A\) gives two right triangles, as in the previous case: \(\triangle ABQ\) and \(\triangle ACQ\). Proceeding as before, we get \(h = b \sin(\gamma)\) and \(h = c \sin(\beta)\) so that \(\frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}\).

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3 Your Science teachers should thank us for this.
4 Don’t worry! Radians will be back before you know it!
Dropping an altitude from vertex B also generates two right triangles, $\triangle ABQ$ and $\triangle BCQ$. We know that $\sin(\alpha') = \frac{h'}{c}$ so that $h' = c \sin(\alpha')$. Since $\alpha' = 180^\circ - \alpha$, $\sin(\alpha') = \sin(\alpha)$, so in fact, we have $h' = c \sin(\alpha)$. Proceeding to $\triangle BCQ$, we get $\sin(\gamma) = \frac{h'}{a}$ so $h' = a \sin(\gamma)$. Putting this together with the previous equation, we get $\sin(\gamma)\frac{c}{a} = \sin(\alpha)\frac{a}{c}$, and we are finished with this case.

The remaining case is when $\triangle ABC$ is a right triangle. In this case, the Law of Sines reduces to the formulas given in Theorem 10.4 and is left to the reader. In order to use the Law of Sines to solve a triangle, we need at least one angle-side opposite pair. The next example showcases some of the power, and the pitfalls, of the Law of Sines.

**Example 11.2.2.** Solve the following triangles. Give exact answers and decimal approximations (rounded to hundredths) and sketch the triangle.

1. $\alpha = 120^\circ$, $a = 7$ units, $\beta = 45^\circ$
2. $\alpha = 85^\circ$, $\beta = 30^\circ$, $c = 5.25$ units
3. $\alpha = 30^\circ$, $a = 1$ units, $c = 4$ units
4. $\alpha = 30^\circ$, $a = 2$ units, $c = 4$ units
5. $\alpha = 30^\circ$, $a = 3$ units, $c = 4$ units
6. $\alpha = 30^\circ$, $a = 4$ units, $c = 4$ units

**Solution.**

1. Knowing an angle-side opposite pair, namely $\alpha$ and $a$, we may proceed in using the Law of Sines. Since $\beta = 45^\circ$, we use $\frac{b}{\sin(45^\circ)} = \frac{7}{\sin(120^\circ)}$ so $b = \frac{7 \sin(45^\circ)}{\sin(120^\circ)} = \frac{7\sqrt{6}}{3} \approx 5.72$ units. Now that we have two angle-side pairs, it is time to find the third. To find $\gamma$, we use the fact that the sum of the measures of the angles in a triangle is $180^\circ$. Hence, $\gamma = 180^\circ - 120^\circ - 45^\circ = 15^\circ$. To find $c$, we have no choice but to used the derived value $\gamma = 15^\circ$, yet we can minimize the propagation of error here by using the given angle-side opposite pair $(\alpha, a)$. The Law of Sines gives us $\frac{c}{\sin(15^\circ)} = \frac{7}{\sin(120^\circ)}$ so that $c = \frac{7 \sin(15^\circ)}{\sin(120^\circ)} \approx 2.09$ units.

2. In this example, we are not immediately given an angle-side opposite pair, but as we have the measures of $\alpha$ and $\beta$, we can solve for $\gamma$ since $\gamma = 180^\circ - 85^\circ - 30^\circ = 65^\circ$. As in the previous example, we are forced to use a derived value in our computations since the only

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5The exact value of $\sin(15^\circ)$ could be found using the difference identity for sine or a half-angle formula, but that becomes unnecessarily messy for the discussion at hand. Thus “exact” here means $\frac{7 \sin(15^\circ)}{\sin(120^\circ)}$. 

---
angle-side pair available is \((\gamma, c)\). The Law of Sines gives \(\frac{a}{\sin(85^\circ)} = \frac{5.25\sin(85^\circ)}{\sin(65^\circ)}\). After the usual rearrangement, we get \(a = \frac{5.25\sin(85^\circ)}{\sin(65^\circ)} \approx 5.77\) units. To find \(b\) we use the angle-side pair \((\gamma, c)\) which yields \(\frac{b}{\sin(30^\circ)} = \frac{5.25\sin(30^\circ)}{\sin(65^\circ)}\) hence \(b = \frac{5.25\sin(30^\circ)}{\sin(65^\circ)} \approx 2.90\) units.

3. Since we are given \((\alpha, a)\) and \(c\), we use the Law of Sines to find the measure of \(\gamma\). We start with \(\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{2}\) and get \(\sin(\gamma) = 4\sin(30^\circ) = 2\). Since the range of the sine function is \([-1, 1]\), there is no real number with \(\sin(\gamma) = 2\). Geometrically, we see that side \(a\) is just too short to make a triangle. The next three examples keep the same values for the measure of \(\alpha\) and the length of \(c\) while varying the length of \(a\). We will discuss this case in more detail after we see what happens in those examples.

4. In this case, we have the measure of \(\alpha = 30^\circ\), \(a = 2\) and \(c = 4\). Using the Law of Sines, we get \(\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{2}\) so \(\sin(\gamma) = 2\sin(30^\circ) = 1\). Now \(\gamma\) is an angle in a triangle which also contains \(\alpha = 30^\circ\). This means that \(\gamma\) must measure between \(0^\circ\) and \(150^\circ\) in order to fit inside the triangle with \(\alpha\). The only angle that satisfies this requirement and has \(\sin(\gamma) = 1\) is \(\gamma = 90^\circ\). In other words, we have a right triangle. We find the measure of \(\beta\) to be \(\beta = 180^\circ - 30^\circ - 90^\circ = 60^\circ\) and then determine \(b\) using the Law of Sines. We find \(b = \frac{2\sin(60^\circ)}{\sin(30^\circ)} = 2\sqrt{3} \approx 3.46\) units. In this case, the side \(a\) is precisely long enough to form a unique right triangle.

5. Proceeding as we have in the previous two examples, we use the Law of Sines to find \(\gamma\). In this case, we have \(\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{3}\) or \(\sin(\gamma) = \frac{4\sin(30^\circ)}{3} = \frac{2}{3}\). Since \(\gamma\) lies in a triangle with \(\alpha = 30^\circ\),
we must have that $0^\circ < \gamma < 150^\circ$. There are two angles $\gamma$ that fall in this range and have 
$
sin(\gamma) = \frac{2}{3}$: $\gamma = \arcsin\left(\frac{2}{3}\right)$ radians $\approx 41.81^\circ$ and $\gamma = \pi - \arcsin\left(\frac{2}{3}\right)$ radians $\approx 138.19^\circ$. At this point, we pause to see if it makes sense that we actually have two viable cases to consider. As we have discussed, both candidates for $\gamma$ are ‘compatible’ with the given angle-side pair $(\alpha, a) = (30^\circ, 3)$ in that both choices for $\gamma$ can fit in a triangle with $\alpha$ and both have a sine of $\frac{2}{3}$. The only other given piece of information is that $c = 4$ units. Since $c > a$, it must be true that $\gamma$, which is opposite $c$, has greater measure than $\alpha$ which is opposite $a$. In both cases, $\gamma > \alpha$, so both candidates for $\gamma$ are compatible with this last piece of given information as well. Thus have two triangles on our hands. In the case $\gamma = \arcsin\left(\frac{2}{3}\right)$ radians $\approx 41.81^\circ$, we find $\beta \approx 180^\circ - 30^\circ - 41.81^\circ = 108.19^\circ$. Using the Law of Sines with the angle-side opposite pair $(\alpha, a)$ and $\beta$, we find $b \approx \frac{3 \sin(108.19^\circ)}{\sin(30^\circ)} \approx 5.70$ units. In the case $\gamma = \pi - \arcsin\left(\frac{2}{3}\right)$ radians $\approx 138.19^\circ$, we repeat the exact same steps and find $\beta \approx 11.81^\circ$ and $b \approx 1.23$ units. Both triangles are drawn below.

6. For this last problem, we repeat the usual Law of Sines routine to find that $\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{4}$ so that $\sin(\gamma) = \frac{1}{2}$. Since $\gamma$ must inhabit a triangle with $\alpha = 30^\circ$, we must have $0^\circ < \gamma < 150^\circ$. Since the measure of $\gamma$ must be strictly less than $150^\circ$, there is just one angle which satisfies both required conditions, namely $\gamma = 30^\circ$. So $\beta = 180^\circ - 30^\circ - 30^\circ = 120^\circ$ and, using the Law of Sines one last time, $b = \frac{4 \sin(120^\circ)}{\sin(30^\circ)} = 4\sqrt{3} \approx 6.93$ units.

Some remarks about Example 11.2.2 are in order. We first note that if we are given the measures of two of the angles in a triangle, say $\alpha$ and $\beta$, the measure of the third angle $\gamma$ is uniquely

\[\gamma = \arcsin\left(\frac{2}{3}\right) - \frac{\pi}{2}\text{ radians and }180^\circ - \arcsin\left(\frac{2}{3}\right)\text{ radians. Hence, }\beta = \pi - \frac{\pi}{2} - \arcsin\left(\frac{2}{3}\right) = \frac{\pi}{2} - \arcsin\left(\frac{2}{3}\right)\text{ radians }\approx 108.19^\circ.\]

\[\beta = \arcsin\left(\frac{2}{3}\right) - \frac{\pi}{3}\text{ radians }\approx 11.81^\circ.\]
determined using the equation $\gamma = 180^\circ - \alpha - \beta$. Knowing the measures of all three angles of a triangle completely determines its shape. If in addition we are given the length of one of the sides of the triangle, we can then use the Law of Sines to find the lengths of the remaining two sides to determine the size of the triangle. Such is the case in numbers 1 and 2 above. In number 1, the given side is adjacent to just one of the angles – this is called the ‘Angle-Angle-Side’ (AAS) case. In number 2, the given side is adjacent to both angles which means we are in the so-called ‘Angle-Side-Angle’ (ASA) case. If, on the other hand, we are given the measure of just one of the angles in the triangle along with the length of two sides, only one of which is adjacent to the given angle, we are in the ‘Angle-Side-Side’ (ASS) case. In number 3, the length of the one given side $a$ was too short to even form a triangle; in number 4, the length of $a$ was just long enough to form a right triangle; in 5, $a$ was long enough, but not too long, so that two triangles were possible; and in number 6, side $a$ was long enough to form a triangle but too long to swing back and form two. These four cases exemplify all of the possibilities in the Angle-Side-Side case which are summarized in the following theorem.

**Theorem 11.3.** Suppose $(\alpha, a)$ and $(\gamma, c)$ are intended to be angle-side pairs in a triangle where $\alpha$, $a$ and $c$ are given. Let $h = c \sin(\alpha)$

- If $a < h$, then no triangle exists which satisfies the given criteria.
- If $a = h$, then $\gamma = 90^\circ$ so exactly one (right) triangle exists which satisfies the criteria.
- If $h < a < c$, then two distinct triangles exist which satisfy the given criteria.
- If $a \geq c$, then $\gamma$ is acute and exactly one triangle exists which satisfies the given criteria.

Theorem 11.3 is proved on a case-by-case basis. If $a < h$, then $a < c \sin(\alpha)$. If a triangle were to exist, the Law of Sines would have $\frac{\sin(\gamma)}{c} = \frac{\sin(\alpha)}{a}$ so that $\sin(\gamma) = \frac{c \sin(\alpha)}{a} > \frac{a}{a} = 1$, which is impossible. In the figure below, we see geometrically why this is the case.

Simply put, if $a < h$ the side $a$ is too short to connect to form a triangle. This means if $a \geq h$, we are always guaranteed to have at least one triangle, and the remaining parts of the theorem

\[\text{If this sounds familiar, it should. From high school Geometry, we know there are four congruence conditions for triangles: Angle-Angle-Side (AAS), Angle-Side-Angle (ASA), Side-Angle-Side (SAS) and Side-Side-Side (SSS). If we are given information about a triangle that meets one of these four criteria, then we are guaranteed that exactly one triangle exists which satisfies the given criteria.} \]

\[\text{In more reputable books, this is called the ‘Side-Side-Angle’ or SSA case.}\]
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tell us what kind and how many triangles to expect in each case. If \( a = h \), then \( a = c \sin(\alpha) \) and the Law of Sines gives \( \frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c} \) so that \( \sin(\gamma) = \frac{c \sin(\alpha)}{a} = \frac{a}{a} = 1 \). Here, \( \gamma = 90^\circ \) as required.

Moving along, now suppose \( h < a < c \). As before, the Law of Sines gives \( \sin(\gamma) = \frac{c \sin(\alpha)}{a} \). Since \( h < a \), \( c \sin(\alpha) < a \) or \( \frac{c \sin(\alpha)}{a} < 1 \) which means there are two solutions to \( \sin(\gamma) = \frac{c \sin(\alpha)}{a} \); an acute angle which we’ll call \( \gamma_0 \), and its supplement, \( 180^\circ - \gamma_0 \). We need to argue that each of these angles ‘fit’ into a triangle with \( \alpha \). Since \((\alpha, a)\) and \((\gamma_0, c)\) are angle-side opposite pairs, the assumption \( c > a \) in this case gives us \( \gamma_0 > \alpha \). Since \( \gamma_0 \) is acute, we must have that \( \alpha \) is acute as well. This means one triangle can contain both \( \alpha \) and \( \gamma_0 \), giving us one of the triangles promised in the theorem. If we manipulate the inequality \( \gamma_0 > \alpha \) a bit, we have \( 180^\circ - \gamma_0 < 180^\circ - \alpha \) which gives \( (180^\circ - \gamma_0) + \alpha < 180^\circ \). This proves a triangle can contain both of the angles \( \alpha \) and \( (180^\circ - \gamma_0) \), giving us the second triangle predicted in the theorem. To prove the last case in the theorem, we assume \( a \geq c \). Then \( \alpha \geq \gamma \), which forces \( \gamma \) to be an acute angle. Hence, we get only one triangle in this case, completing the proof.

---

One last comment before we use the Law of Sines to solve an application problem. In the Angle-Side-Side case, if you are given an obtuse angle to begin with then it is impossible to have the two triangle case. Think about this before reading further.

**Example 11.2.3.** Sasquatch Island lies off the coast of Ippizuti Lake. Two sightings, taken 5 miles apart, are made to the island. The angle between the shore and the island at the first observation point is \( 30^\circ \) and at the second point the angle is \( 45^\circ \). Assuming a straight coastline, find the distance from the second observation point to the island. What point on the shore is closest to the island? How far is the island from this point?

**Solution.** We sketch the problem below with the first observation point labeled as \( P \) and the second as \( Q \). In order to use the Law of Sines to find the distance \( d \) from \( Q \) to the island, we first need to find the measure of \( \beta \) which is the angle opposite the side of length 5 miles. To that end, we note that the angles \( \gamma \) and \( 45^\circ \) are supplemental, so that \( \gamma = 180^\circ - 45^\circ = 135^\circ \). We can now find \( \beta = 180^\circ - 30^\circ - \gamma = 180^\circ - 30^\circ - 135^\circ = 15^\circ \). By the Law of Sines, we have \( \frac{d}{\sin(30^\circ)} = \frac{5}{\sin(15^\circ)} \) which gives \( d = \frac{5\sin(30^\circ)}{\sin(15^\circ)} \approx 9.66 \) miles. Next, to find the point on the coast closest to the island, which we’ve labeled as \( C \), we need to find the perpendicular distance from the island to the coast.\(^{\text{11}}\)

---

\(^{\text{10}}\)Remember, we have already argued that a triangle exists in this case!

\(^{\text{11}}\)Do you see why \( C \) must lie to the right of \( Q \)?
Let \( x \) denote the distance from the second observation point \( Q \) to the point \( C \) and let \( y \) denote the distance from \( C \) to the island. Using Theorem 10.4, we get \( \sin(45^\circ) = \frac{y}{d} \). After some rearranging, we find \( y = d \sin(45^\circ) \approx 9.66 \left( \frac{\sqrt{2}}{2} \right) \approx 6.83 \) miles. Hence, the island is approximately 6.83 miles from the coast. To find the distance from \( Q \) to \( C \), we note that \( \beta = 180^\circ - 90^\circ - 45^\circ = 45^\circ \) so by symmetry,\(^{12} \) we get \( x = y \approx 6.83 \) miles. Hence, the point on the shore closest to the island is approximately 6.83 miles down the coast from the second observation point.

\[
\begin{align*}
\text{Sasquatch Island} & \quad \text{Shoreline} & \quad \text{Sasquatch Island} \\
\text{P} & \quad \beta & \quad \beta \\
\gamma & \quad 45^\circ & \quad 45^\circ \\
30^\circ & \quad d \approx 9.66 \text{ miles} & \quad y \text{ miles} \\
5 \text{ miles} & \quad \text{Q} & \quad x \text{ miles} \\
\end{align*}
\]

We close this section with a new formula to compute the area enclosed by a triangle. Its proof uses the same cases and diagrams as the proof of the Law of Sines and is left as an exercise.

**Theorem 11.4.** Suppose \((\alpha, a),(\beta, b)\) and \((\gamma, c)\) are the angle-side opposite pairs of a triangle. Then the area \( A \) enclosed by the triangle is given by

\[
A = \frac{1}{2}bc \sin(\alpha) = \frac{1}{2}ac \sin(\beta) = \frac{1}{2}ab \sin(\gamma)
\]

**Example 11.2.4.** Find the area of the triangle in Example 11.2.2 number 1.

**Solution.** From our work in Example 11.2.2 number 1, we have all three angles and all three sides to work with. However, to minimize propagated error, we choose \( A = \frac{1}{2}ac \sin(\beta) \) from Theorem 11.4 because it uses the most pieces of given information. We are given \( a = 7 \text{ and } \beta = 45^\circ \), and we calculated \( c = \frac{7 \sin(15^\circ)}{\sin(120^\circ)} \). Using these values, we find \( A = \frac{1}{2}(7) \left( \frac{7 \sin(15^\circ)}{\sin(120^\circ)} \right) \sin(45^\circ) \approx 5.18 \) square units. The reader is encouraged to check this answer against the results obtained using the other formulas in Theorem 11.4.

\(^{12}\) Or by Theorem 10.4 again . . .
11.3 The Law of Cosines

In Section 11.2, we developed the Law of Sines (Theorem 11.2) to enable us to solve triangles in the ‘Angle-Angle-Side’ (AAS), the ‘Angle-Side-Angle’ (ASA) and the ambiguous ‘Angle-Side-Side’ (ASS) cases. In this section, we develop the Law of Cosines which handles solving triangles in the ‘Side-Angle-Side’ (SAS) and ‘Side-Side-Side’ (SSS) cases.1 We state and prove the theorem below.

**Theorem 11.5. Law of Cosines:** Given a triangle with angle-side opposite pairs \((\alpha, a)\), \((\beta, b)\) and \((\gamma, c)\), the following equations hold:

\[
\begin{align*}
    a^2 &= b^2 + c^2 - 2bc \cos(\alpha) \\
    b^2 &= a^2 + c^2 - 2ac \cos(\beta) \\
    c^2 &= a^2 + b^2 - 2ab \cos(\gamma)
\end{align*}
\]

or, solving for the cosine in each equation, we have:

\[
\begin{align*}
    \cos(\alpha) &= \frac{b^2 + c^2 - a^2}{2bc} \\
    \cos(\beta) &= \frac{a^2 + c^2 - b^2}{2ac} \\
    \cos(\gamma) &= \frac{a^2 + b^2 - c^2}{2ab}
\end{align*}
\]

To prove the theorem, we consider a generic triangle with the vertex of angle \(\alpha\) at the origin with side \(b\) positioned along the positive \(x\)-axis.

From this set-up, we immediately find that the coordinates of \(A\) and \(C\) are \(A(0, 0)\) and \(C(b, 0)\). From Theorem 10.3, we know that since the point \(B(x, y)\) lies on a circle of radius \(c\), the coordinates

---

1Here, ‘Side-Angle-Side’ means that we are given two sides and the ‘included’ angle - that is, the given angle is adjacent to both of the given sides.
of \( B \) are \( B(x, y) = B(c \cos(\alpha), c \sin(\alpha)) \). (This would be true even if \( \alpha \) were an obtuse or right angle so although we have drawn the case when \( \alpha \) is acute, the following computations hold for any angle \( \alpha \) drawn in standard position where \( 0 < \alpha < 180^\circ \).) We note that the distance between the points \( B \) and \( C \) is none other than the length of side \( a \). Using the distance formula, Equation 1.1, we get

\[
a = \sqrt{(c \cos(\alpha) - b)^2 + (c \sin(\alpha) - 0)^2}
\]

\[
a^2 = \left( \sqrt{(c \cos(\alpha) - b)^2 + c^2 \sin^2(\alpha)} \right)^2
\]

\[
a^2 = (c \cos(\alpha) - b)^2 + c^2 \sin^2(\alpha)
\]

\[
a^2 = c^2 \cos^2(\alpha) - 2bc \cos(\alpha) + b^2 + c^2 \sin^2(\alpha)
\]

\[
a^2 = c^2 (1 + b^2 - 2bc \cos(\alpha))
\]

Since \( \cos^2(\alpha) + \sin^2(\alpha) = 1 \)

The remaining formulas given in Theorem 11.5 can be shown by simply reorienting the triangle to place a different vertex at the origin. We leave these details to the reader. What’s important about \( a \) and \( \alpha \) in the above proof is that \((\alpha, a)\) is an angle-side opposite pair and \( b \) and \( c \) are the sides adjacent to \( \alpha \) – the same can be said of any other angle-side opposite pair in the triangle. Notice that the proof of the Law of Cosines relies on the distance formula which has its roots in the Pythagorean Theorem. That being said, the Law of Cosines can be thought of as a generalization of the Pythagorean Theorem. If we have a triangle in which \( \gamma = 90^\circ \), then \( \cos(\gamma) = \cos(90^\circ) = 0 \) so we get the familiar relationship \( c^2 = a^2 + b^2 \). What this means is that in the larger mathematical sense, the Law of Cosines and the Pythagorean Theorem amount to pretty much the same thing.\(^2\)

**Example 11.3.1.** Solve the following triangles. Give exact answers and decimal approximations (rounded to hundredths) and sketch the triangle.

1. \( \beta = 50^\circ, \ a = 7 \) units, \( c = 2 \) units
2. \( a = 4 \) units, \( b = 7 \) units, \( c = 5 \) units

**Solution.**

1. We are given the lengths of two sides, \( a = 7 \) and \( c = 2 \), and the measure of the included angle, \( \beta = 50^\circ \). With no angle-side opposite pair to use, we apply the Law of Cosines. We get \( b^2 = 7^2 + 2^2 - 2(7)(2) \cos(50^\circ) \) which yields \( b = \sqrt{53 - 28 \cos(50^\circ)} \approx 5.92 \) units. In order to determine the measures of the remaining angles \( \alpha \) and \( \gamma \), we are forced to use the derived value for \( b \). There are two ways to proceed at this point. We could use the Law of Cosines again, or, since we have the angle-side opposite pair \((\beta, b)\) we could use the Law of Sines. The advantage to using the Law of Cosines over the Law of Sines in cases like this is that unlike the sine function, the cosine function distinguishes between acute and obtuse angles. The cosine of an acute is positive, whereas the cosine of an obtuse angle is negative. Since the sine of both acute and obtuse angles are positive, the sine of an angle alone is not

---

\(^2\)This shouldn’t come as too much of a shock. All of the theorems in Trigonometry can ultimately be traced back to the definition of the circular functions along with the distance formula and hence, the Pythagorean Theorem.
enough to determine if the angle in question is acute or obtuse. Since both authors of the textbook prefer the Law of Cosines, we proceed with this method first. When using the Law of Cosines, it’s always best to find the measure of the largest unknown angle first, since this will give us the obtuse angle of the triangle if there is one. Since the largest angle is opposite the longest side, we choose to find $\alpha$ first. To that end, we use the formula $\cos(\alpha) = \frac{b^2 + c^2 - a^2}{2bc}$ and substitute $a = 7$, $b = \sqrt{53 - 28 \cos(50^\circ)}$ and $c = 2$. We get

$$\cos(\alpha) = \frac{2 - 7 \cos(50^\circ)}{\sqrt{53 - 28 \cos(50^\circ)}}$$

Since $\alpha$ is an angle in a triangle, we know the radian measure of $\alpha$ must lie between 0 and $\pi$ radians. This matches the range of the arccosine function, so we have

$$\alpha = \arccos \left( \frac{2 - 7 \cos(50^\circ)}{\sqrt{53 - 28 \cos(50^\circ)}} \right) \text{ radians} \approx 114.99^\circ$$

At this point, we could find $\gamma$ using $\gamma = 180^\circ - \alpha - \beta \approx 180^\circ - 114.99^\circ - 50^\circ = 15.01^\circ$, that is if we trust our approximation for $\alpha$. To minimize propagation of error, however, we could use the Law of Cosines again,\(^4\) in this case using $\cos(\gamma) = \frac{a^2 + b^2 - c^2}{2ab}$. Plugging in $a = 7$, $b = \sqrt{53 - 28 \cos(50^\circ)}$ and $c = 2$, we get $\gamma = \arccos \left( \frac{7 - 2 \cos(50^\circ)}{\sqrt{53 - 28 \cos(50^\circ)}} \right) \text{ radians} \approx 15.01^\circ$. We sketch the triangle below.

![Triangle Sketch]

As we mentioned earlier, once we’ve determined $b$ it is possible to use the Law of Sines to find the remaining angles. Here, however, we must proceed with caution as we are in the ambiguous (ASS) case. It is advisable to first find the smallest of the unknown angles, since we are guaranteed it will be acute.\(^5\) In this case, we would find $\gamma$ since the side opposite $\gamma$ is smaller than the side opposite the other unknown angle, $\alpha$. Using the angle-side opposite pair $(\beta, b)$, we get $\frac{\sin(\gamma)}{2} = \frac{\sin(50^\circ)}{\sqrt{53 - 28 \cos(50^\circ)}}$. The usual calculations produces $\gamma \approx 15.01^\circ$ and $\alpha = 180^\circ - \beta - \gamma \approx 180^\circ - 50^\circ - 15.01^\circ = 114.99^\circ$.

2. Since all three sides and no angles are given, we are forced to use the Law of Cosines. Following our discussion in the previous problem, we find $\beta$ first, since it is opposite the longest side, $b$. We get $\cos(\beta) = \frac{a^2 + c^2 - b^2}{2ac} = -\frac{1}{5}$, so we get $\beta = \arccos \left( -\frac{1}{5} \right) \text{ radians} \approx 101.54^\circ$. As in

\(^3\)after simplifying . . .

\(^4\)Your instructor will let you know which procedure to use. It all boils down to how much you trust your calculator.

\(^5\)There can only be one obtuse angle in the triangle, and if there is one, it must be the largest.
the previous problem, now that we have obtained an angle-side opposite pair \((\beta, b)\), we could proceed using the Law of Sines. The Law of Cosines, however, offers us a rare opportunity to find the remaining angles using only the data given to us in the statement of the problem. Using this, we get \(\gamma = \arccos \left( \frac{5}{7} \right)\) radians \(\approx 44.42^\circ\) and \(\alpha = \arccos \left( \frac{29}{35} \right)\) radians \(\approx 34.05^\circ\).

We note that, depending on how many decimal places are carried through successive calculations, and depending on which approach is used to solve the problem, the approximate answers you obtain may differ slightly from those the authors obtain in the Examples and the Exercises. A great example of this is number 2 in Example 11.3.1, where the approximate values we record for the measures of the angles sum to 180.01°, which is geometrically impossible. Next, we have an application of the Law of Cosines.

**Example 11.3.2.** A researcher wishes to determine the width of a vernal pond as drawn below. From a point \(P\), he finds the distance to the eastern-most point of the pond to be 950 feet, while the distance to the western-most point of the pond from \(P\) is 1000 feet. If the angle between the two lines of sight is 60°, find the width of the pond.

**Solution.** We are given the lengths of two sides and the measure of an included angle, so we may apply the Law of Cosines to find the length of the missing side opposite the given angle. Calling this length \(w\) (for width), we get \(w^2 = 950^2 + 1000^2 - 2(950)(1000)\cos(60^\circ) = 952500\) from which we get \(w = \sqrt{952500} \approx 976\) feet. \(\square\)
In Section 11.2, we used the proof of the Law of Sines to develop Theorem 11.4 as an alternate formula for the area enclosed by a triangle. In this section, we use the Law of Cosines to derive another such formula - Heron’s Formula.

**Theorem 11.6. Heron’s Formula:** Suppose $a$, $b$ and $c$ denote the lengths of the three sides of a triangle. Let $s$ be the semiperimeter of the triangle, that is, let $s = \frac{1}{2}(a + b + c)$. Then the area $A$ enclosed by the triangle is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

We prove Theorem 11.6 using Theorem 11.4. Using the convention that the angle $\gamma$ is opposite the side $c$, we have $A = \frac{1}{2}ab\sin(\gamma)$ from Theorem 11.4. In order to simplify computations, we start by manipulating the expression for $A^2$.

$$A^2 = \left(\frac{1}{2}ab\sin(\gamma)\right)^2$$

$$= \frac{1}{4}a^2b^2\sin^2(\gamma)$$

$$= \frac{a^2b^2}{4} \left(1 - \cos^2(\gamma)\right) \quad \text{since} \quad \sin^2(\gamma) = 1 - \cos^2(\gamma).$$

The Law of Cosines tells us $\cos(\gamma) = \frac{a^2+b^2-c^2}{2ab}$, so substituting this into our equation for $A^2$ gives

$$A^2 = \frac{a^2b^2}{4} \left(1 - \cos^2(\gamma)\right)$$

$$= \frac{a^2b^2}{4} \left[1 - \left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right]$$

$$= \frac{a^2b^2}{4} \left[1 - \left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right]$$

$$= \frac{a^2b^2}{4} \left[4a^2b^2 - (a^2+b^2-c^2)^2\right]$$

$$= \frac{4a^2b^2 - (a^2+b^2-c^2)^2}{16}$$

$$= \frac{(2ab)^2 - (a^2+b^2-c^2)^2}{16}$$

$$= \frac{(2ab - (a^2+b^2-c^2))(2ab + (a^2+b^2-c^2))}{16}$$

$$= \frac{(c^2 - a^2 + 2ab - b^2)(a^2 + 2ab + b^2 - c^2)}{16}$$

difference of squares.
\[ A^2 = \frac{(c^2 - [a^2 - 2ab + b^2]) ([a^2 + 2ab + b^2] - c^2)}{16} \]
\[ = \frac{(c^2 - (a - b)^2) ((a + b)^2 - c^2)}{16} \]
\[ = \frac{(c - (a - b))(c + (a - b))((a + b) - c)((a + b) + c)}{16} \]
\[ = \frac{(b + c - a)(a + c - b)(a + b - c)(a + b + c)}{16} \]
\[ = \frac{(b + c - a)}{2} \cdot \frac{(a + c - b)}{2} \cdot \frac{(a + b - c)}{2} \cdot \frac{(a + b + c)}{2} \]

At this stage, we recognize the last factor as the semiperimeter, \( s = \frac{1}{2}(a + b + c) = \frac{a+b+c}{2} \). To complete the proof, we note that
\[
(s - a) = \frac{a + b + c}{2} - a = \frac{a + b + c - 2a}{2} = \frac{b + c - a}{2}
\]
Similarly, we find \( (s - b) = \frac{a+c-b}{2} \) and \( (s - c) = \frac{a+b-c}{2} \). Hence, we get
\[
A^2 = \frac{(b + c - a)}{2} \cdot \frac{(a + c - b)}{2} \cdot \frac{(a + b - c)}{2} \cdot \frac{(a + b + c)}{2}
\]
\[ = (s - a)(s - b)(s - c)s \]
so that \( A = \sqrt{s(s - a)(s - b)(s - c)} \) as required.

We close with an example of Heron’s Formula.

**Example 11.3.3.** Find the area enclosed of the triangle in Example 11.3.1 number 2.

**Solution.** We are given \( a = 4 \), \( b = 7 \) and \( c = 5 \). Using these values, we find \( s = \frac{1}{2}(4 + 7 + 5) = 8 \), \( (s - a) = 8 - 4 = 4 \), \( (s - b) = 8 - 7 = 1 \) and \( (s - c) = 8 - 5 = 3 \). Using Heron’s Formula, we get \( A = \sqrt{s(s - a)(s - b)(s - c)} = \sqrt{(8)(4)(1)(3)} = \sqrt{96} = 4\sqrt{6} \approx 9.80 \) square units.
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