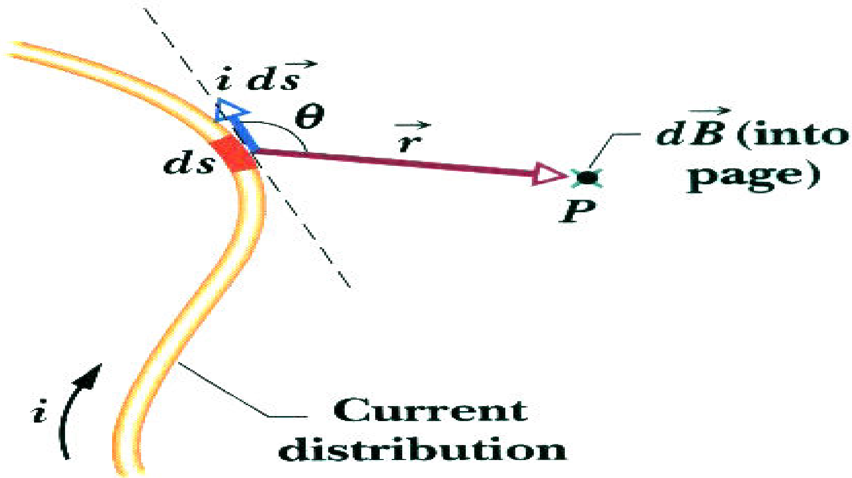
Magnetic Fields Due to Currents

In the early $1820^{s,}$ the magnetic deflection of a compass needle in close proximity to a current carrying wire was observed. An empirical result, known as the **Biot-Savart Law**, giving the magnitude and direction of the magnetic field followed shortly afterwards.

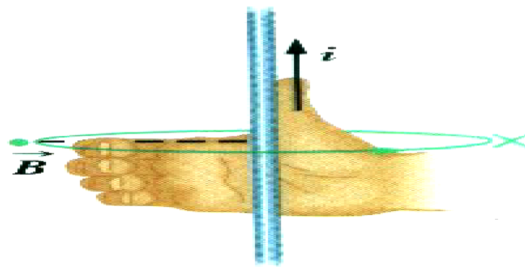


Here, $i\overrightarrow{ds}$ is a differential current-length element producing a differential magnetic field $\overrightarrow{dB}$ at location P. Summing over all such current-length elements will give the magnetic field $\vec{B}$ using superposition. The Biot-Savart Law for the magnetic field is:

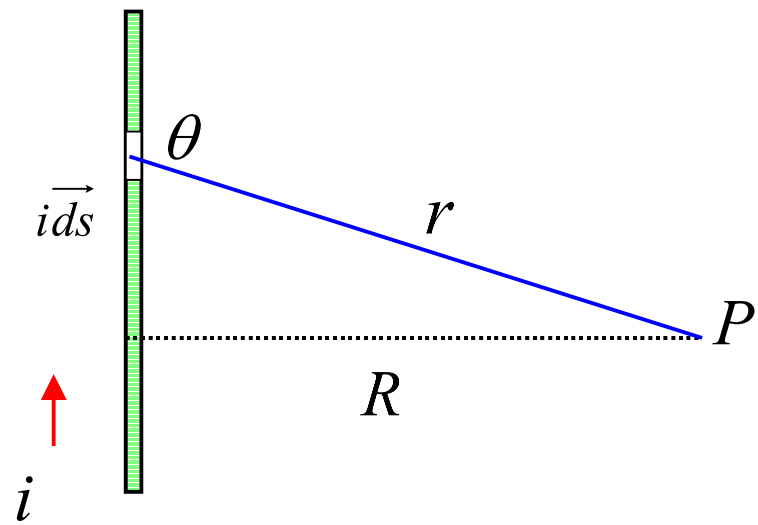
$$\overrightarrow{dB} = \frac{\mu_0}{4\pi}\frac{i\overrightarrow{ds}\times\vec{r}}{r^3} \qquad\qquad |\overrightarrow{dB}| = \frac{\mu_0}{4\pi}\frac{idsSin(\theta)}{r^2}$$

$$\mu_0 = Permeability\_of\_Free\_Space = 4\pi\times10^{-7}\frac{Tm}{A}$$

Note $1/r^2$ dependence and **RHR** direction into the page in the above figure. Placing the thumb along the current-length direction, then the fingers curl into the field direction.



E.g., Infinite Current Carrying Wire



$$\vec{B} = 2 \times \frac{\mu_0}{4\pi} \int_0^{\infty} i ds \frac{Sin(\theta)}{r^2}$$ Into the page at **P.**

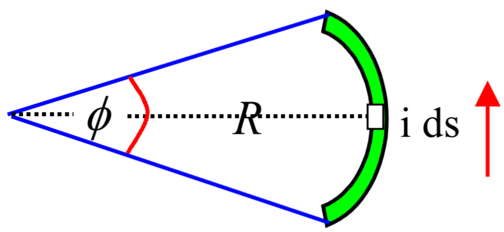
Using $Sin(\theta) = R \Big/ \sqrt{R^2 + s^2}$

$$\vec{B} = \frac{\mu_0 i}{2\pi} \int_0^{\infty} \frac{R}{\left(R^2 + s^2\right)^{3/2}} ds$$ Into the page at **P.**

$$B=\frac{\mu_0 i}{2\pi}\frac{Rs}{R^2\left(s^2+R^2\right)^{1/2}}\Bigg|_0^{\infty}=\frac{\mu_0 i}{2\pi R}$$

For a **semi-infinite** wire then, $$B=\frac{\mu_0 i}{4\pi R}$$

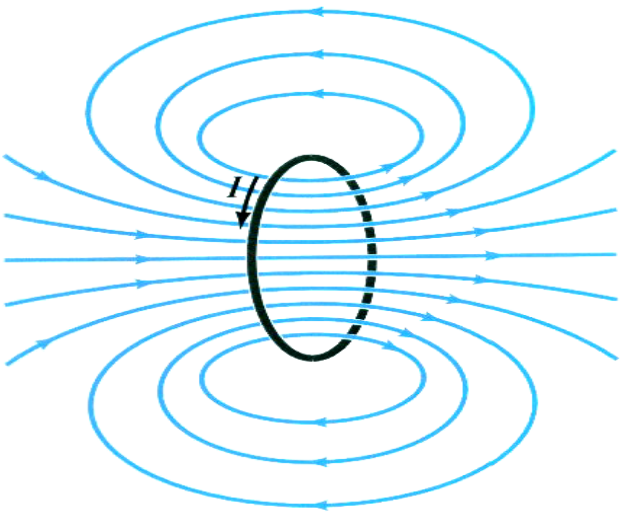
E.g., Arc Length of Current Carrying Wire

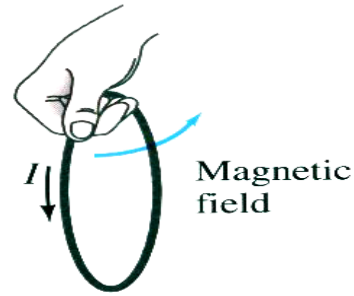


Here the variable θ in Biot-Savart is always 90^0

$$B=\frac{\mu_0}{4\pi}\int_0^{\phi} i\frac{Rd\phi}{R^2}=\frac{\mu_0 i\phi}{4\pi R}$$

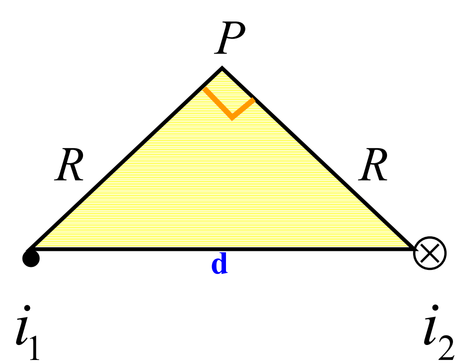
$$\vec{B}=2\pi\times\frac{\mu_0 i}{4\pi R}=\frac{\mu_0 i}{2R}$$ For a full loop of wire.

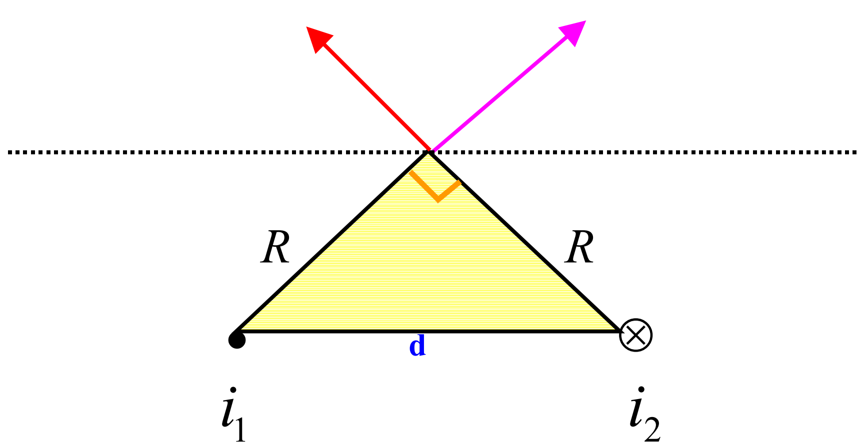




E.g., A Superposition Problem

Given the geometry shown, find the magnetic field at point P due to i_1 and i_2





$$| B_1 | = \frac{\mu_0 i_1}{2\pi R}$$

$$| B_2 | = \frac{\mu_0 i_2}{2\pi R}$$

X and Y components of each of these vectors may be found from the geometry and a superposition formed.

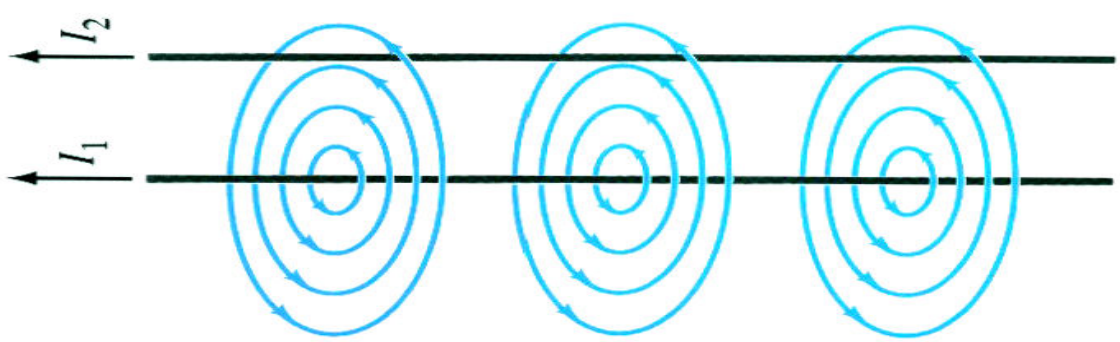
$R = dCos(45^0)$ And the answer is in terms of **d.**

Force between Two Parallel Current Carrying Wires

For moving charges in a magnetic field, there exists a force: $\vec{F} = q\vec{v} \times \vec{B}$

The force on a current carrying element in a magnetic field is: $\overrightarrow{dF} = i\overrightarrow{dl} \times \vec{B}$

If we have then two parallel current-carrying wires as shown, then the field established by current **1** at the location of wire **2** results in a force on wire **2** and vice-versa.



For a perpendicular separation distance of **d,** the field from current **1** at a distance **d** is:

$$| B_1 | = \frac{\mu_0 I_1}{2\pi d}$$

Since the direction of this field is at **90 degrees** to $\mathbf{I_2 l_2}$ the force on wire **2** is:

$$F_{21} = \frac{\mu_0 I_1 I_2 L}{2\pi d}$$

Consideration of the cross products and the RHR indicates that:

Parallel currents in the same direction ➔ an attractive force between the two wires.

Parallel currents in opposing directions ➔ a repulsive force between the two wires.

From this force also comes the SI base quantity the **Ampere**. The ampere is defined as that amount of current, the same in each wire, yielding $\frac{F_{21}}{L} = 2x10^{-7} N/m$ at a distance of 1-meter. Note from this μ_0 has the value $\mu_0 = 4\pi \times 10^{-7} \frac{N}{A^2}$

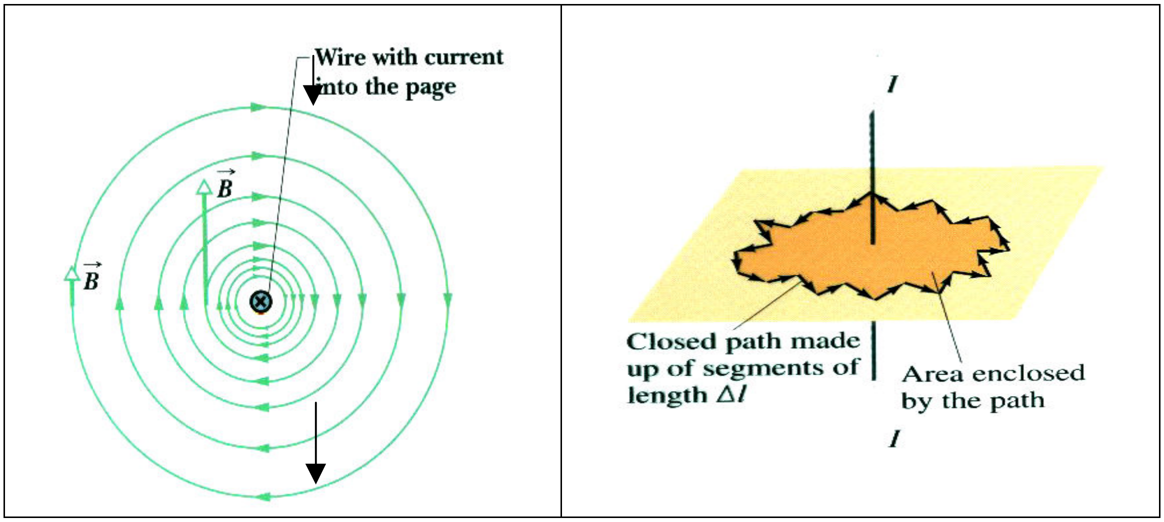
Ampere's Law

Andre' Ampere, again in the early 1820s, discovered a specific connection between **steady currents** and the resulting magnetic field, which is particularly suited to current distributions possessing a high degree of symmetry.

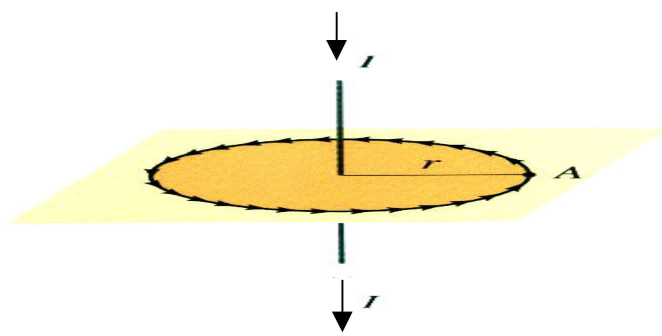
The result known as Ampere's Law considers a closed **Amperian loop** path integral about the current distribution and relates the evaluated integral to the **arithmetic sum** of all currents enclosed within the loop:

$$\oint \vec{B} \cdot \vec{ds} = \mu_0 i_{enclosed}$$

For example, recall the magnetic field of a long straight current-carrying wire looks like



The dot product $\overrightarrow{B} \cdot \overrightarrow{ds}$ means that field components perpendicular to the path differential element do not contribute to the integral. For a tractable integrand, the path might be more judiciously chosen in accordance with the problem symmetry as circular.



By convention, the **direction chosen for the path integration** determines the arithmetic sign +/- of the current contributions to the RHS of Ampere's Law.

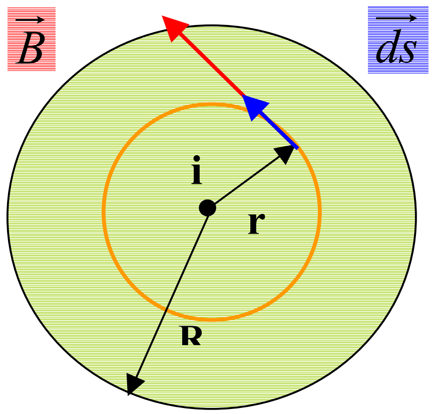
In the above example, a counterclockwise path integration ➔ positive current will be in the direction of ones thumb after curling the fingers of the right hand in the direction of integration. The current shown, therefore, will give a minus current in Ampere's Law.

$$\oint \overrightarrow{B} \cdot \overrightarrow{ds} = -\mu_0 I$$

Here if we solve for the field, $B = \dfrac{+\mu_0 I}{2\pi R}$ since $\vec{B}$ & $\overrightarrow{ds}$ are at 180 degrees.

E.g., Magnetic Field inside a Long Straight Wire

The magnetic field within a wire carrying a current i may be evaluated with Ampere's Law by determining $i_{enclosed}$ at the interior point of evaluation.

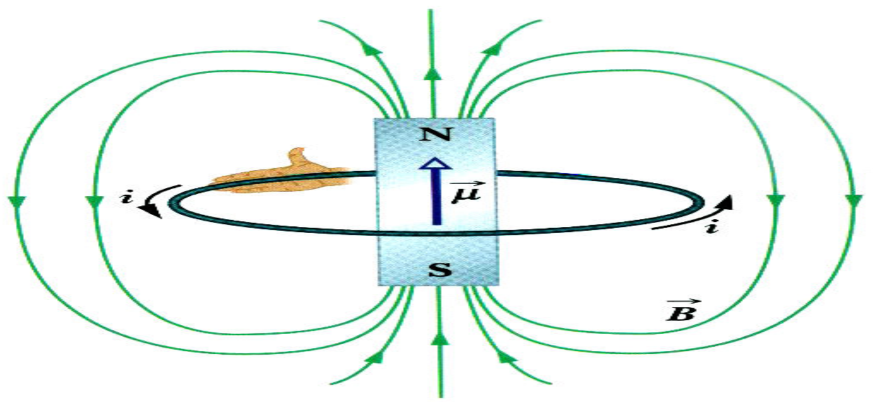


$$i_{enclosed} = i * \frac{\pi r^2}{\pi R^2} \quad \rightarrow \quad B = \frac{1}{2\pi r} * \mu_0 i * \frac{\pi r^2}{\pi R^2}$$

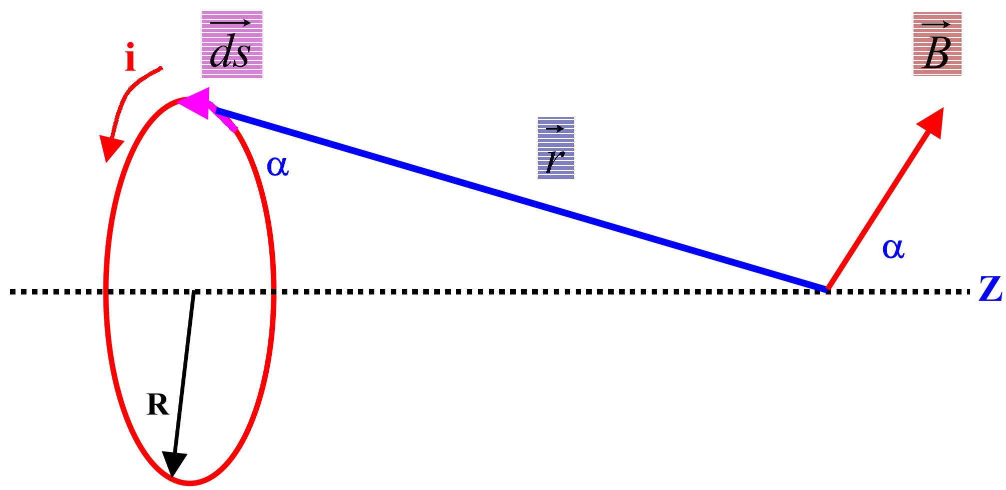
$$B_{Inside} = \mu_0 i \frac{r}{2\pi R^2}$$

Current Carrying Coil as a Magnetic Dipole

The magnetic field due to a current loop is similar to that from a bar magnet and forms a magnetic dipole moment as shown.



The field on-axis due to the current loop may be evaluated using the Biot-Savart Law



Components of $\vec{B}$ that are in the vertical cancel by symmetry.

The field parallel to **Z** is $|\overrightarrow{dB_{//}}| = dBCos(\alpha)$

From Biot-Savart, $$dB = \frac{\mu_0 i}{4\pi}\frac{dsSin(90)}{r^2}$$

From the figure, $$Cos(\alpha) = \frac{R}{\sqrt{R^2 + z^2}}$$

$$|\vec{B}| = \int dB_{//} = \frac{\mu_0 iR}{4\pi\left(R^2 + z^2\right)^{3/2}} \int_0^{2\pi} ds$$

$$|\vec{B}| = \frac{\mu_0 iR^2}{2\left(R^2 + z^2\right)^{3/2}}$$

Note the $\mathbf{Z = 0}$ (center of loop) result and the large $\mathbf{Z}$ limit $\propto \mathbf{Z^{-3}}$

In this large $\mathbf{Z}$ limit and for $\mathbf{N}$ loops of area $A = \pi R^2$

$$|\vec{B}| = \frac{\mu_0 NiA}{2\pi z^3} = \frac{\mu_0}{2\pi z^3}|\vec{\mu}|$$

$$\vec{\mu} = NiA \qquad RHR$$

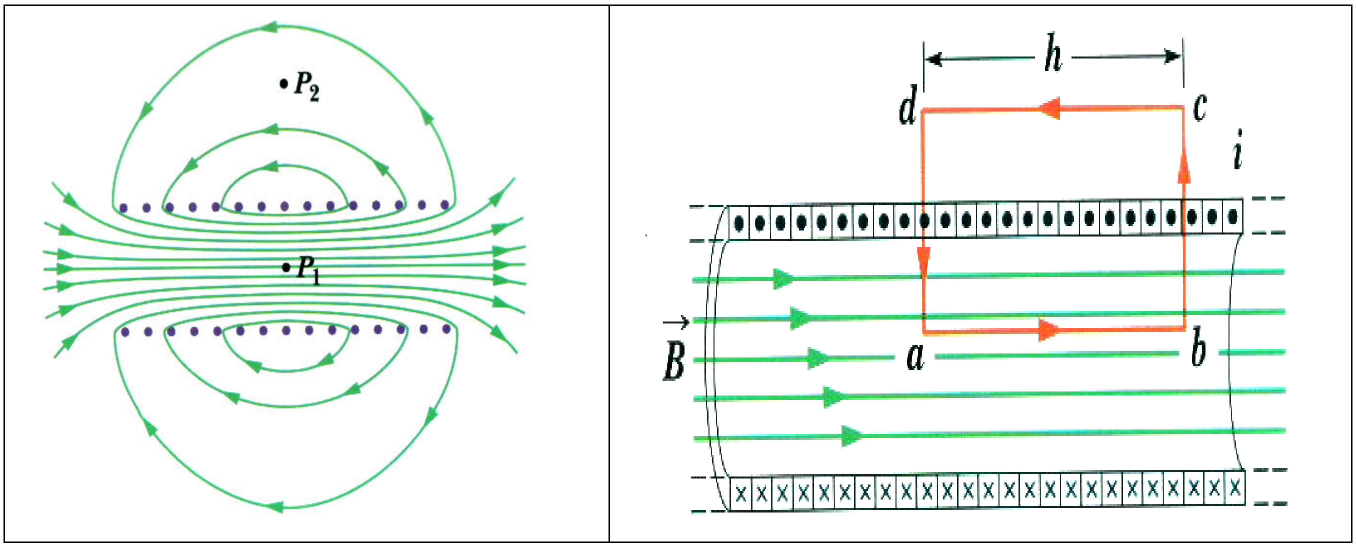
Current loops therefore appear as magnetic dipoles at large $\mathbf{Z}$ and $\mu = NiA$

Solenoids

A **solenoid** consists of N tightly packed current carrying loops in a linear configuration. The magnetic field is a superposition of the fields from each individual current loop.

Ideal solenoids have an infinite # of loops, uniform field inside, and zero field outside.

Application of Ampere's Law to this problem determines the on-axis magnetic field as:



$$\oint \vec{B} \cdot \vec{ds} = \int_a^b \vec{B} \cdot \vec{ds} + \int_b^c \vec{B} \cdot \vec{ds} + \int_c^d \vec{B} \cdot \vec{ds} + \int_d^a \vec{B} \cdot \vec{ds} = \mu_0 i_{enclosed}$$

$$i_{enclosed} = i * nh \qquad nh = \frac{number\_of\_turns}{length} * length$$

The only contribution to the path integration is along **a→b.**

$$Bh = \mu_0 inh \quad \rightarrow \quad B_{Ideal\_Solenoid} = \mu_0 ni$$

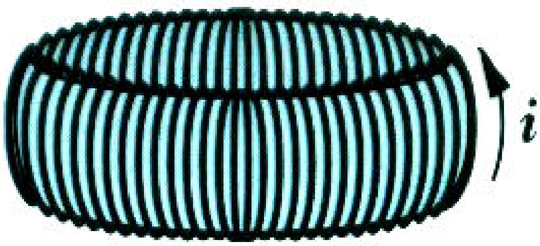
Toroids

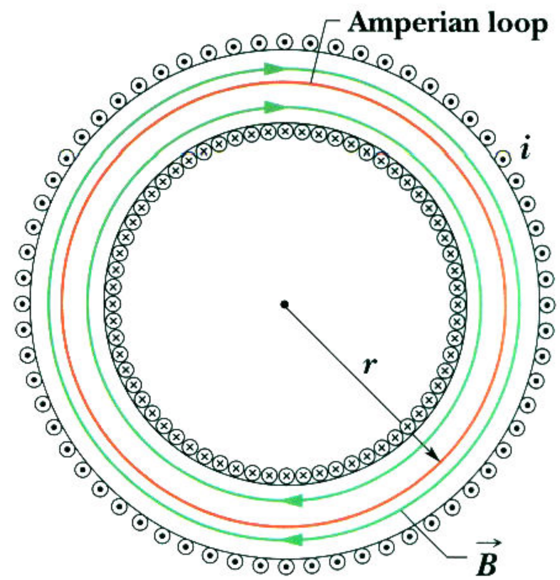
A **toroid** consists of N tightly packed current carrying loops in a doughnut configuration.

The magnetic field is a superposition of the fields from each individual current loop.

Ideal toroids have non-uniform magnetic field inside, and zero magnetic field outside.

Application of Ampere's Law to this problem determines the interior magnetic field as:





$$\oint \vec{B} \cdot \vec{ds} = \mu_0 N i \qquad N = number\_of\_turns$$

$$B * 2\pi r = \mu_0 N i$$

$$B_{Toroid} = \frac{\mu_0 N i}{2\pi r}$$ Note this is not constant over the toroid cross-section.

Magnetic Materials

Three types of magnetic materials categorized according to their magnetic nature are:

1) **Diamagnetic**
2) **Paramagnetic**
3) **Ferromagnetic**

Beginning with a definition of **magnetization** $\vec{M}$ for a material placed in an external magnetic field $\vec{B}_0$, evaluate the material response to this external field.

$$\vec{M} = N\vec{\mu}/V = density\_of\_magnetic\_dipole\_moments$$

$$\vec{\mu} = magnetic\_dipole\_moment\_for\_1\_atom$$

Units of $\vec{M}$ $\qquad A \cdot m^2 / m^3 = A/m$

The **net magnetic field** within the material is then a superposition of $\mu_0 \vec{M}$ with $\vec{B}_0$

$$\vec{B} = \vec{B}_0 + \mu_0 \vec{M} = \vec{B}_0 + \chi_m \vec{B}_0$$

$$\chi_m = magnetic\_susceptibility$$

$$\mu = \mu_0(1 + \chi_m) \qquad \mu = magnetic\_permeability$$

In the vacuum, $\mu = \mu_0 \qquad \chi_m = 0$

Diamagnetic case,

$$\mu < \mu_0 \qquad \chi_m < 0 \qquad \vec{M} \quad opposite \quad \vec{B_0}$$

Paramagnetic case,

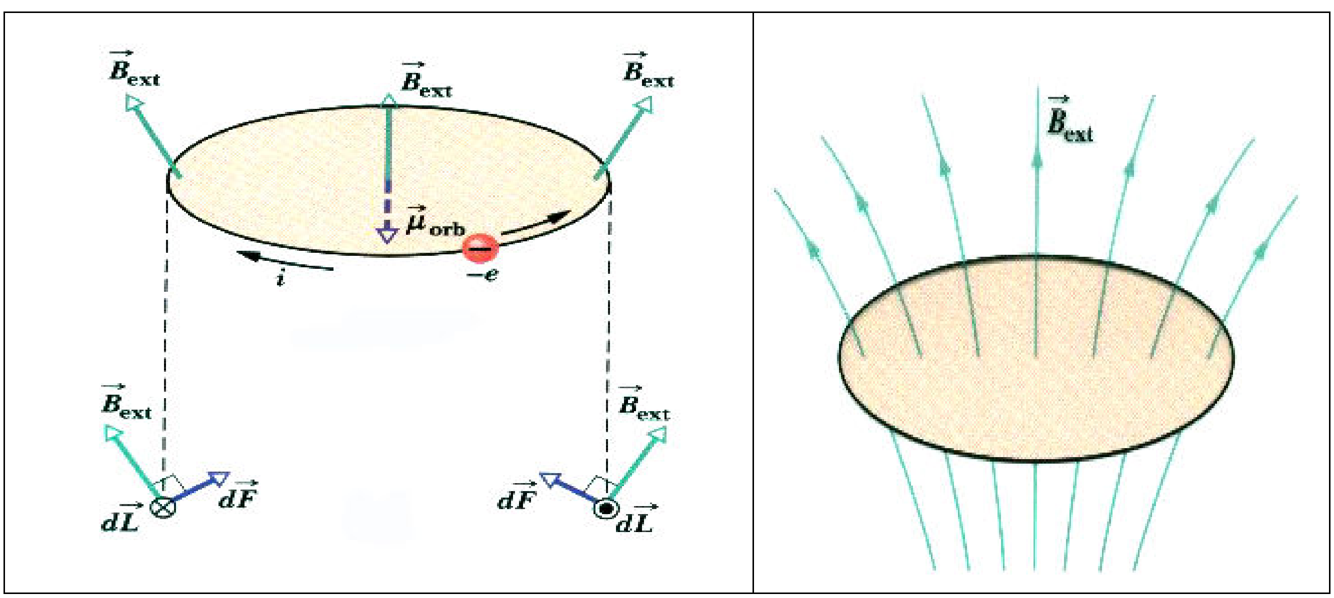
$$\mu > \mu_0 \qquad \chi_m > 0 \qquad \vec{M} \quad aligned\_with \quad \vec{B_0}$$

Ferromagnetic case, $\mu >> \mu_0 \qquad \chi_m >> 1$

Diamagnetism

Materials in which each atom has a zero net electronic magnetic dipole moment throughout the volume are diamagnetic.

Placing these materials in a non-uniform external magnetic field will induce oppositely directed magnetic dipole moments in the material and result in a small repulsive force away from high external field regions.



In diamagnetic materials $K_m = \mu / \mu_0 = relative\_permeability$ is approximately unity implying the response of the material to an external field is small and does not persist as $\vec{B}_0 \to 0$

Typical diamagnetic **magnetic susceptibilities** $\chi_m = K_m - 1$ are on the order of 10^{-6}

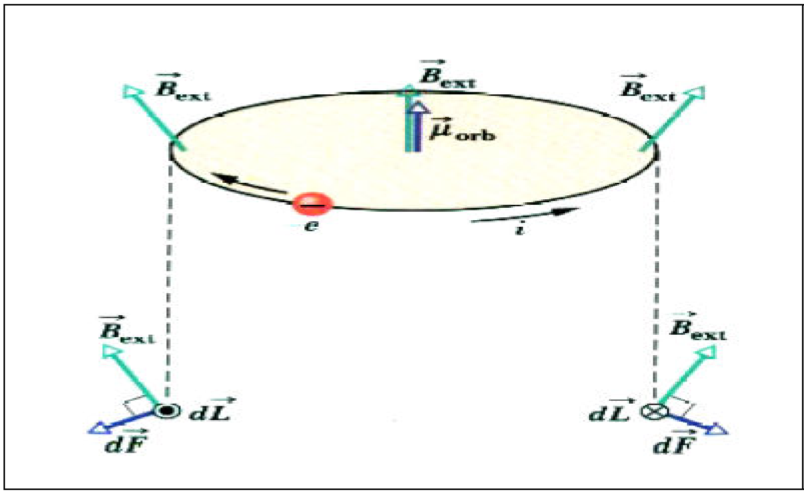
Some diamagnetic substances are:

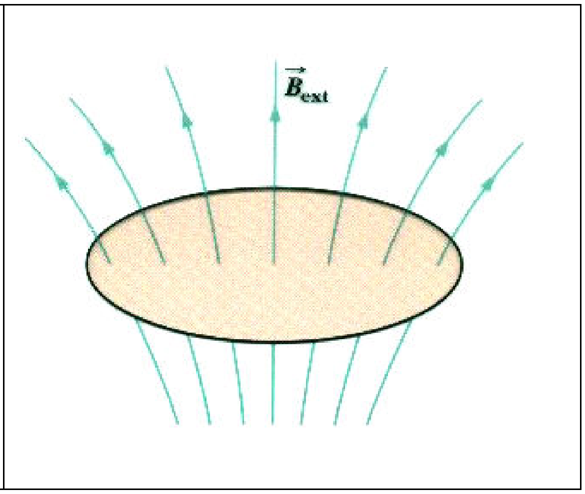
- Bismuth
- Copper
- Lead
- Silicon
- Diamond

Paramagnetism

Materials in which each atom has a net magnetic dipole moment, but these moments are not aligned largely throughout the volume are paramagnetic.

Placing these materials in a non-uniform external magnetic field will align the preexisting magnetic dipole moments with the external field direction and result in a small attractive force toward high external field regions.





In paramagnetic materials $K_m = \mu / \mu_0 = relative\_permeability$ is approx. unity implying the response of the material to an external field is small and does not persist as $\vec{B}_0 \rightarrow 0$

Typical paramagnetic **magnetic susceptibilities** $\chi_m = K_m - 1$ are on the order of 10^{-5}

Some paramagnetic substances are:

- Aluminum
- Calcium
- Oxygen (STP)
- Platinum
- Tungsten

Paramagnetic materials have a magnetization proportional to the applied external field according to **Curie's Law**: Holding true for $\mu B_0 / kT << 1$

$$M = C\frac{B_0}{T} \qquad C = Curie's\_Const. \qquad T\_in\_Kelvin$$

Ferromagnetism

Materials in which each atom has a net magnetic dipole moment, and alignment randomizing thermal effects are secondary to spin-spin **("exchange coupling")** magnetic dipole moment aligning interactions, are ferromagnetic.

In ferromagnetic materials a **magnetic domain** structure exist where the alignment of magnetic dipole moments within any given **domain boundary** is largely coherent.

Each domain region has a large net magnetic dipole moment vector that orients independently of neighboring domains such that bulk samples aren't initially magnetized.

Placing these materials in a non-uniform external magnetic field will either align **domain** magnetic dipole moments with the external field direction or grow in size domains with moments collinear to $\vec{B}_0$. An attractive force toward high external field regions results.

For a ferromagnet, $\chi_m >> 1$ implies the response of the material to external fields is large and persists as $\vec{B}_0 \rightarrow 0$ ➔ **permanent magnetization and hysteresis**

Typical ferromagnetic **magnetic susceptibilities** $\chi_m = K_m - 1$ are the order of 10^3

Some ferromagnetic substances are:

- Magnetic Iron
- Nickel
- Cobalt
- Permalloy (Ni 78.5%; Fe 21.5%)
- Mumetal (Ni 75%; Fe 18%; Cu 5%; Cr 2%)

Large χ_m implies a small external applied magnetic field will produce a large magnetic field within the ferromagnetic material

Ferromagnetism is lost if the sample temperature is raised above the **Curie Temperature**

At this point, a ferromagnetic material becomes paramagnetic and permanent magnetism will disappear.

For iron, $T_C = 770^0 C$