Capacitance

The assemblage of electric charges into a finite region of space requires work done by an external agent equal to the increase in electric potential energy.

Increased energy resides in the electric field established as the charges are assembled.

We have seen with other forms of potential energy, such as gravitational or chemical for example, that a capability of converting potential energy into other energy forms is of considerable importance.

Controlling **electric potential energy** such that it may be converted into other forms of energy useful in application depends on how precisely we can assemble electric charges.

Using the fact that excess electric charge resides on the surface of a conductor, and if we use a conductor as our finite region of space onto which we assemble electric charge, then it is the **geometry of the conductor** that determines how charges are positioned.

Two isolated conductors, in the same system, on which $\pm Q$ electric charge is maintained creates a potential difference between the conductors and an electric field.

Capacitors are **passive circuit elements** that store or release electric charge as needed.

Typically consisting of two 'small' conductors separated by a gap filled with air or some other material, **capacitance** value refers to how much $\pm Q$ electric charge needs to be placed on these conductors to produce a **1-volt** potential difference across the gap.

Capacitance is measured in **Farads**: $1Farad = {}^{1\_Coulomb}\!/_{1\_Volt}$

A capacitor rated at **1-farad** means that if **1-coulomb** of charge were positioned on the conducting elements of the capacitor, then a potential difference of **1-volt** would be produced across the separation gap.

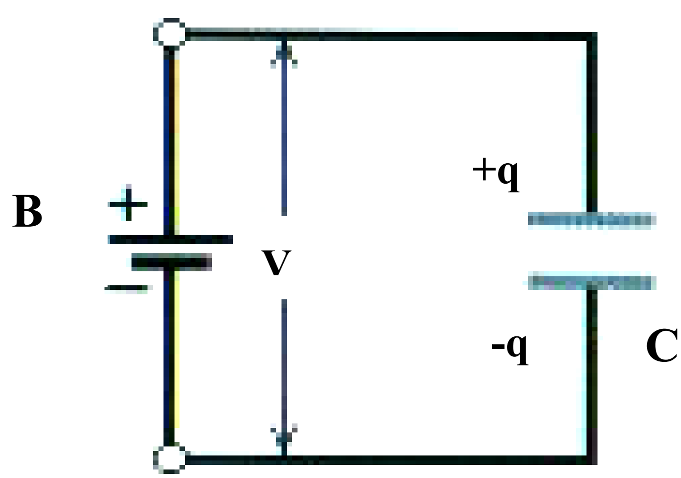
Larger capacitance →more $\pm Q$ **required for the same** ΔV .

Recall the **permitivity of free space** $\varepsilon_0 = 8.85x10^{-12} \dfrac{C^2}{N \cdot m^2}$

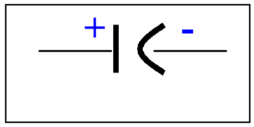
Since the Volt is a Joule/Coulomb, $\varepsilon_0 = 8.85x10^{-12} \dfrac{F}{m}$

$$Q = CV \quad Where \quad V \quad is \quad \Delta V \quad between \quad the \quad conductors.$$

A simple circuit showing a battery **B** and the circuit symbol for a capacitor **C** is:



A capacitor of specific **polarity** is one which must be oriented in a circuit with the + / - leads of the capacitor connected to the + / - terminals of the battery respectively. Its circuit symbol is usually shown with a 'curved' conductor at the negative plate:



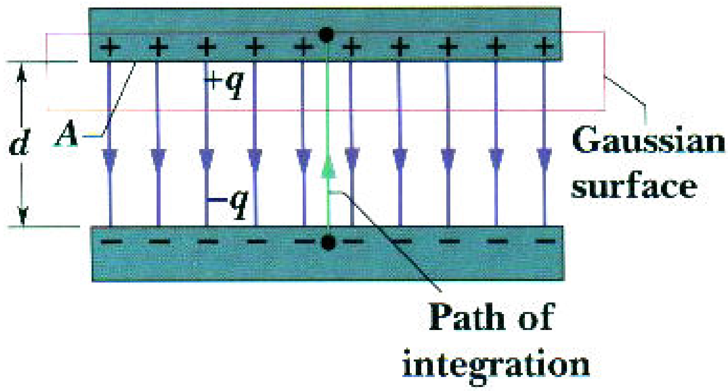
A potential difference is created between the battery + / - terminals by chemical reactions in the battery. Once attached to the capacitor uncharged conductors, electrons are drawn away from the + plate leaving it with a net positive charge and electrons are forced onto the - plate leaving this conductor with an equal and opposite negative charge.

As the capacitor fully charges, the potential difference between its plates matches the potential of the battery ➔ no more current since zero ΔV➔ zero $\vec{E}$

The capacitor remains in a charged state until it is discharged through a resistive circuit or by leakage.

With Gauss' Law $\oint \vec{E} \bullet \hat{n} dA = q_{enclosed} / \varepsilon_0$ and the equation relating electric potential to electric field $\Delta V = -\int_i^f \vec{E} \cdot \vec{dr}$, the capacitance for various conductor geometries may be found as follows:

Parallel Plate Capacitor:



$$\oint \vec{E} \bullet \hat{n}\, dA = {q_{enclosed}} / {\varepsilon_0}$$

$$\vec{E} \, // \, \hat{n} \quad \rightarrow \quad E = {q} / {\varepsilon_0 A}$$

$$\Delta V = -\int_i^f \vec{E} \cdot \vec{ds}$$

$$\vec{E} \nparallel \vec{ds} \quad \rightarrow \quad E = {\Delta V} / {d}$$

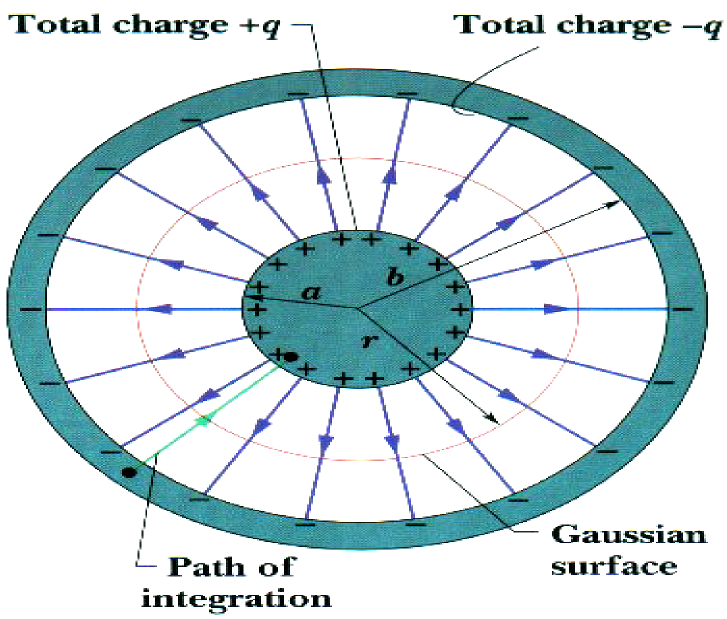
Equating the two expressions for the electric field, we have the capacitance as:

$$C = {q} / {\Delta V} = {\varepsilon_0 A} / {d}$$

Note dependence only on the conductor geometry.

Cylindrical Capacitor:

A cross-sectional end view of this type of capacitor is shown:

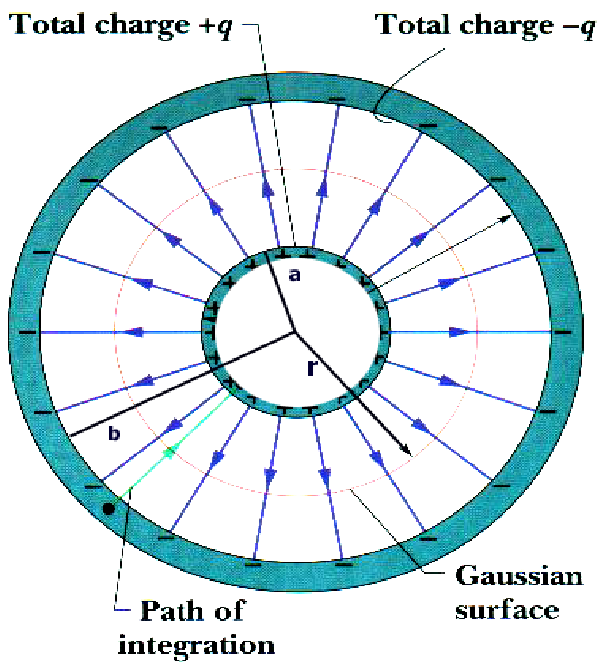


$\oint \vec{E} \bullet \hat{n} dA = q_{enclosed} / \varepsilon_0$	$\vec{E} // \hat{n}$ ➔ $E = q / 2\pi\varepsilon_0 r L$

$$\Delta V = -\int_i^f \vec{E} \cdot \vec{ds} = -\int_a^b \vec{E} \cdot \vec{ds} = -\int_a^b \frac{q}{2\pi\varepsilon_0 r L} Cos(180) dr = \frac{q}{2\pi\varepsilon_0 L} * \ln\frac{b}{a}$$

$$C = q / \Delta V = 2\pi\varepsilon_0 L \Big/ \ln b/a$$

Spherical Shell Capacitor:



$\oint \vec{E} \bullet \hat{n} dA = q_{enclosed} / \varepsilon_0$	$\vec{E} // \hat{n}$ ➔ $E = q / 4\pi\varepsilon_0 r^2$

$$\Delta V = -\int_i^f \vec{E} \cdot \vec{ds} = -\int_a^b \vec{E} \cdot \vec{ds} = -\int_a^b \frac{q}{4\pi\varepsilon_0 r^2} Cos(\pi) dr = \frac{q}{4\pi\varepsilon_0} * \left\{ \frac{1}{a} - \frac{1}{b} \right\}$$

$$C = q / \Delta V = \frac{4\pi\varepsilon_0}{b-a} * ab$$

Charged Isolated Conducting Sphere…Other Conducting Surface Located at ∞:

Capacitance in this case corresponds to the amount of charge in Coulombs that we need to add to the conducting sphere **[radius a]** to increase its voltage **1V.**

$\oint \vec{E} \bullet \hat{n} dA = q_{enclosed} / \varepsilon_0$	$\vec{E} // \hat{n}$ ➔ $E = q / 4\pi\varepsilon_0 r^2$

$$\Delta V = -\int_i^f \vec{E} \cdot \vec{ds} = -\int_a^\infty \vec{E} \cdot \vec{ds} = -\int_a^\infty \frac{q}{4\pi\varepsilon_0 r^2} Cos(\pi) dr = \frac{q}{4\pi\varepsilon_0} * \left\{ \frac{1}{a} - \frac{1}{\infty} \right\}$$

$$C = q / \Delta V = 4\pi\varepsilon_0 a$$

Capacitors in Circuits

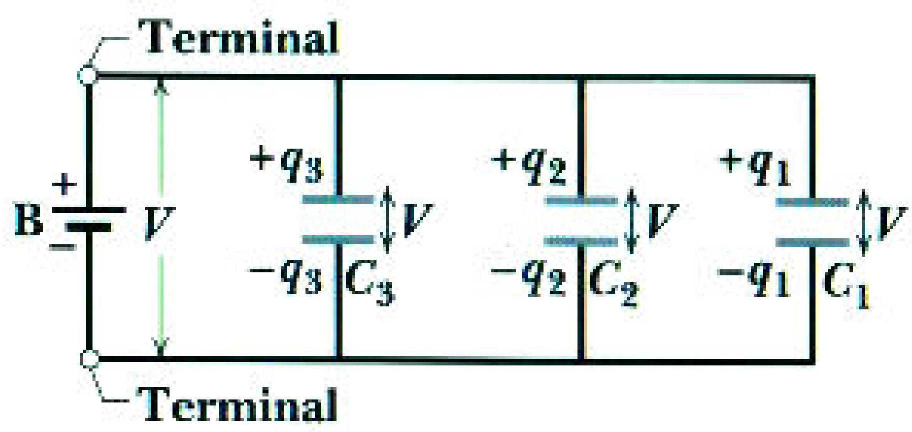
Availability of charge on demand and an ability to store charge for future use is of obvious use in electronics applications where electric charge is the primary commodity.

Circuits containing more than one capacitor may be characterized in terms of an **equivalent capacitance**.

Calculation of equivalent capacitance depends on the circuit lay out and how the capacitors are connected.

Capacitors in Parallel

A parallel connection of capacitor elements is shown.



For capacitors in parallel connections, the **same voltage** is across each element.

The total charge is $q = q_1 + q_2 + q_3$

Using the general relation for the capacitance $q = CV$

$$q = q_1 + q_2 + q_3 = C_1V + C_2V + C_3V$$

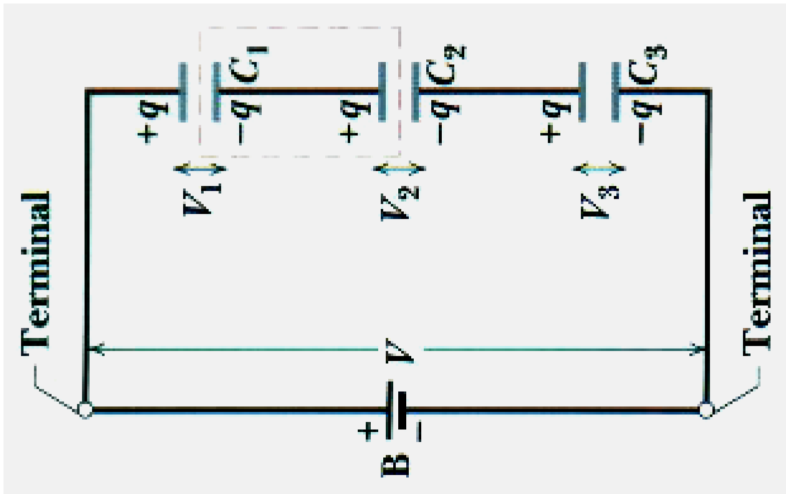
$$q = V\{C_1 + C_2 + C_3\}$$

A circuit of parallel-connected capacitors has an equivalent capacitance

$$C_{eqv} = \sum_{i=1}^{N} C_i$$

Capacitors in Series

A series connection of capacitor elements is shown.



For capacitors in series connections, the **same charge** is across each element.

The voltage drop across each element depends on its capacitance, and the supply voltage is the sum of voltage drops across all the capacitors.

The total voltage is $V = V_1 + V_2 + V_3$

Using the general relation for the capacitance $q = CV$

$$V = V_1 + V_2 + V_3 = \frac{q}{C_1} + \frac{q}{C_2} + \frac{q}{C_3}$$

$$q = C_{eqv} V \Rightarrow C_{eqv} = \left\{ \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \right\}^{-1}$$

Series-connected capacitors have an equivalent capacitance

$$C_{eqv} = \left\{ \sum_{i=1}^{N} \frac{1}{C_i} \right\}^{-1}$$

Note that charge from the battery is directly placed only on the conductors directly wired to the battery. The 'center' conductor is isolated and charge redistributes, as a field is set.

Capacitor Energy

As noted to start, an amount of work is necessary to charge a capacitor and this can be associated with the energy required to establish the electric field:

$$W = \int dW = \int_0^q V dq \qquad W = \int_0^q \frac{q}{C} dq = \frac{q^2}{2C}$$

The electric potential energy of the charged capacitor is:

$$U = \frac{q^2}{2C} = \frac{1}{2} C V^2$$

If a parallel plate capacitor is used as an example, then the **energy density** is:

$$u = \frac{U}{Ad} = \frac{\frac{1}{2} C V^2}{Ad} = \frac{1}{2} \frac{\varepsilon_0 A}{d} * \frac{V^2}{Ad}$$

$$u = \frac{1}{2}\varepsilon_0 E^2$$

In general, given an electric field in some region of space, the electric potential energy per volume in that region is given by $u = \frac{1}{2}\varepsilon_0 E^2$

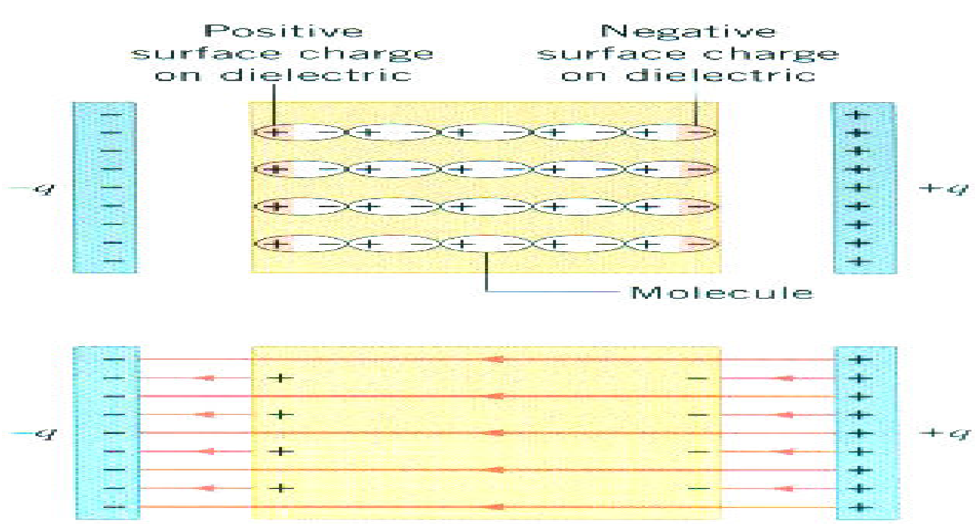
Dielectrics

A **dielectric** is an insulating material that is used to change the capacitance of a capacitor by its insertion between the capacitor conductors.

Polar dielectric materials have molecules with permanent electric dipole moments.

When subjected to the capacitor electric field such moments orient themselves with the electric field, which induces a surface charge on the dielectric surface thereby reducing the overall electric field inside the capacitor and increasing the capacitance:

$$C = \frac{q}{V} = \frac{q}{Ed} \Rightarrow E \downarrow \quad C \uparrow$$



Dielectric materials are characterized by a **dielectric constant κ.**

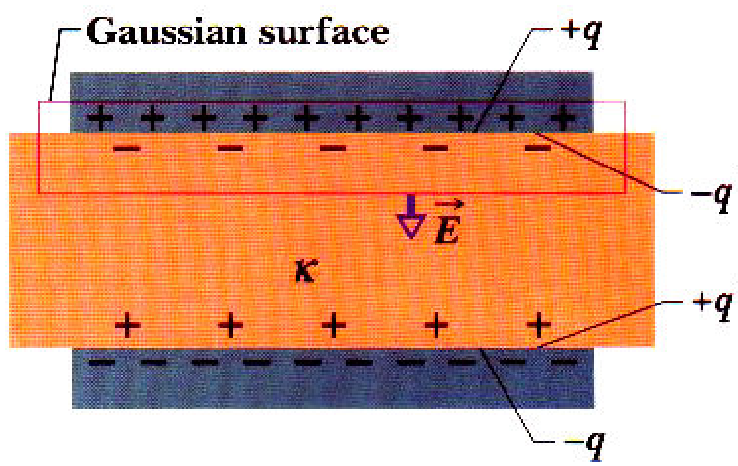
The effect on capacitance is $C = \kappa C_{air}$ $\qquad \kappa \geq 1$

The equations of electrostatics may be modified by letting $\varepsilon_0 \to \kappa\varepsilon_0$

For example, $C = \varepsilon_0 A / d \to \kappa\varepsilon_0 A / d$

Gauss' Law with Dielectrics Present

Using Gauss' Law inside the dielectric filled capacitor the induced surface charges $\pm q'$ need to be taken into account:



$$\oint \vec{E} \bullet \hat{n} dA = \frac{q_{enclosed}}{\varepsilon_0} = \frac{q - q'}{\varepsilon_0}$$

$$E = \frac{q - q'}{\varepsilon_0 A} = \frac{E_0}{\kappa}$$

E_0 is the field strength without the dielectric.

$$q - q' = \varepsilon_0 A \frac{E_0}{\kappa} = \frac{q}{\kappa}$$

Requiring $q' < q$ and $q' = 0$ for $\kappa = 1$.

Gauss' Law is modified:

$$\oint \vec{E} \bullet \hat{n} dA = \frac{q - q'}{\varepsilon_0} = \frac{q}{\kappa \varepsilon_0}$$

$$\oint \kappa \vec{E} \bullet \hat{n} dA = \frac{q}{\varepsilon_0}$$

q *is the free charge*

Current and Resistivity

When considering devices or **circuits** that use electricity to operate, a **biased flow** of **conduction electrons** within or through a **solid-state** material is typically normal for component operation or charging.

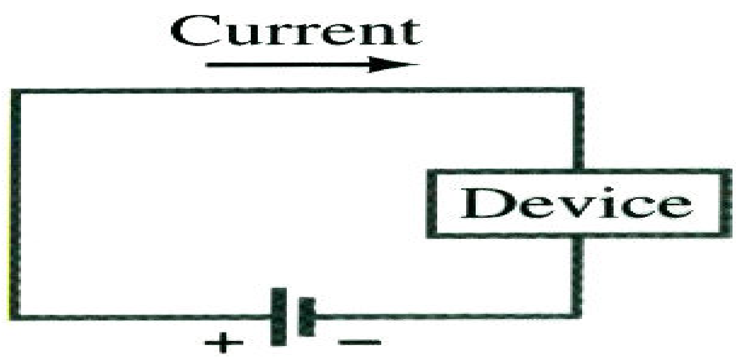
Electric current is defined as the net transport of electric charge through a cross-section area of the medium. **Steady current** implies that the net transport rate is constant.

Resistivity is a material property of the conduction medium indicating the electron current response to an applied electric field.

Different materials will have different electron conduction responses for a given applied electric field depending on the material, copper, iron, silicon, etc.

Biasing electron flow in a circuit is usually done with batteries or power supplies.

A **battery** maintains a potential difference between two terminals $\pm$ by expending internal chemical energy, and provides a steady flow of electrons within the circuit/device to which it is subsequently attached.



a) **An electric field directed away from the + battery terminal and toward the − terminal of the battery is established in a circuit in which terminals are connected.**

b) **Electron current is from the negative terminal, through the circuit, toward the positive terminal.**

c) **Conventional current is taken to be in the direction in which positive charge carriers would move and is therefore in the opposite direction of the electron current.**

d) $$i = \frac{dq}{dt}$$ dq **is charge passing through a cross-section of the conduction medium in an amount of time** dt **.**

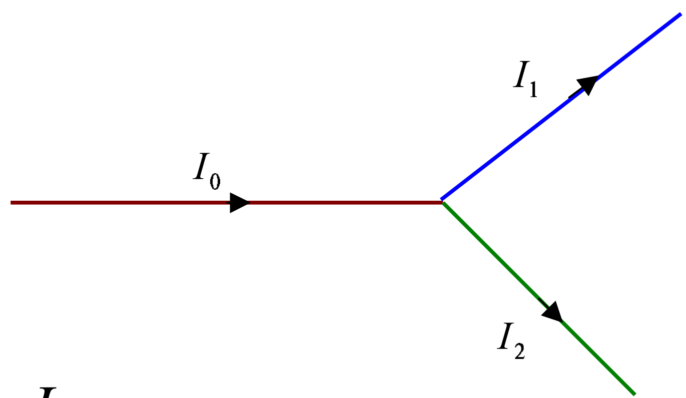
e) **SI units of current is the Ampere: 1A = 1C/s**

f) **[Two 1A currents in wires 1-m apart have a magnetic interaction of 2E-7N]**

Current Junction Law and Current Density

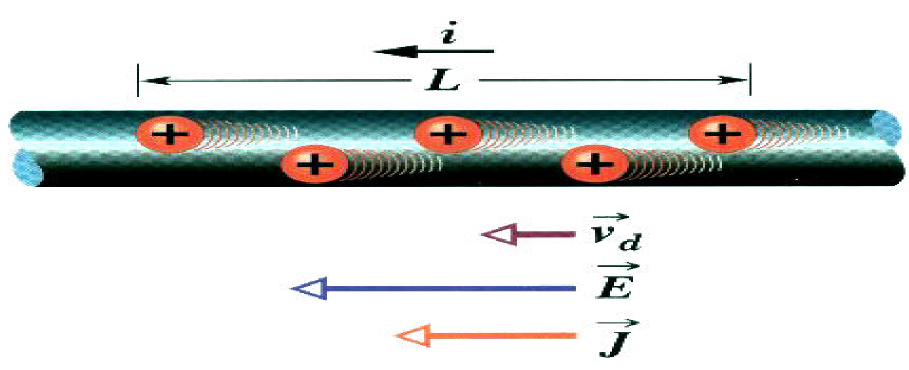
Kirchhoff's 1st Rule for circuits is a statement of the conservation of electric charge at a convergence point or junction within a circuit.

At any junction point, the sum of all currents into that junction must equal the sum of all currents leaving the junction.



$$I_0 = I_1 + I_2$$

Current Density is a vector quantity with magnitude equal to current per cross-section area of the conduction medium. The direction of the current density vector is the same as the conventional current direction.



Using the current density of the conduction medium, the total current may be found as:

$$I_{Total} = \int \vec{J} \cdot \hat{n} dA$$

For uniform current density, $J = \dfrac{I}{A} \quad in \quad Amps/m^2$

The **conduction charge drift speed** $\overrightarrow{v_d}$ may be found in terms of the current density:

For a length of conductor **L**, a transit time of $t = \dfrac{L}{v_d}$ is needed.

The amount of charge pushed through the segment **L** in this time is:

$$q = n(AL)e \quad Where \quad n \quad is \quad the \quad \# \quad of \quad Carriers/Volume$$

The current is $I = \dfrac{q}{t} = nev_d A$

Giving the drift velocity as: $v_d = \dfrac{I}{neA} = \dfrac{J}{ne}$

The vector relation is $\vec{J} = ne\overrightarrow{v_d}$

For + charge carriers ne **is positive and** $\vec{J}$ **is in the same direction as** $\overrightarrow{v_d}$

For − charge carriers ne **is negative and** $\vec{J}$ **is in the opposite direction of** $\overrightarrow{v_d}$

Typical drift velocities are on the order of **10^{-4} m/s**, which is small compared to thermal velocities at **~10^{6} m/s**.

Moving at drift velocity, conduction electrons will need a few hours in order to move a meter. The conduction of electricity, however, is closer to the speed of light as conduction electrons pass along information about any applied electric field.

Resistance and Resistivity

The **resistance** of an object is defined as: $$R = \frac{Voltage\_Applied}{Current\_Established}$$

The SI unit of resistance is the **Ohm**. $$1Ohm = 1\Omega = \frac{1V}{1A}$$

Notice that whereas the resistivity is a material property the does not change given a constant temperature; the measured resistance may vary depending on how the voltage is applied since a **physically different application** may change the current in the object.

Resistivity is a function of material: $$\rho = \frac{Electric\_Field}{Current\_Density} = \frac{E}{J}$$

$$[\rho] = \frac{V/m}{A/m^2} = \Omega \cdot m$$

From dimensions, the relationship between resistance and resistivity may be inferred:

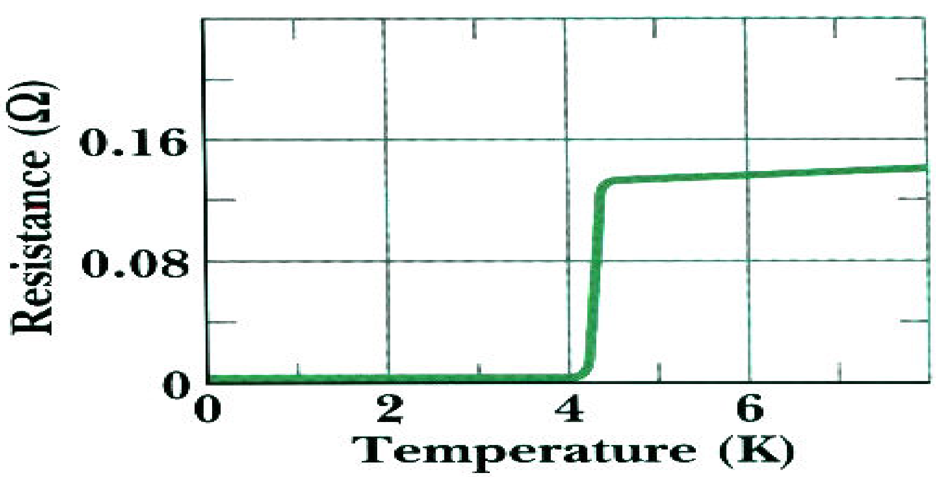
$$\rho = \frac{R/L}{1/A} = \frac{RA}{L} \qquad\qquad R = \rho\frac{L}{A}$$

The **conductivity σ** is defined as the reciprocal of the resistivity:

$$\sigma = \frac{1}{\rho} = \frac{J}{E}$$

Resistivity and Temperature

From Superconductivity, we know that resistance and therefore resistivity exhibits a dependence on sample temperature.



The resistivity of **metals** is found to be approximately linear in temperature variation as:

$$\rho - \rho_0 = \rho_0\alpha(T - T_0)$$

$$\rho_0 = resistivity\_at\_T_0 \qquad T_0\_usually\_293K$$

α **Temperature Coefficient of Resistivity.**

If α is positive, as is the case of conductors, then ρ increases with increasing temperature.

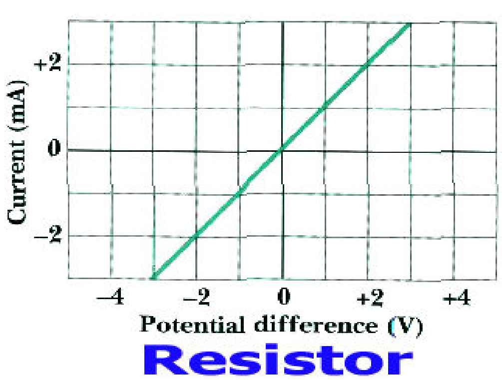
Explanation: The numbers of conduction electron collisions are increasing and thereby interfering with current.

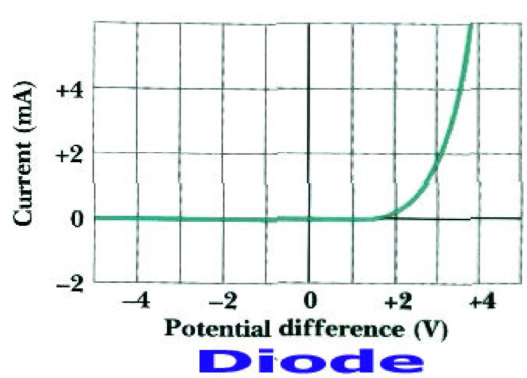
Values of α for semiconductors, however, are negative meaning that ρ decreases with increasing temperature.

Explanation: At higher temperatures in semiconductors, electrons normally not contributing to the conduction process in a semiconductor now have enough energy to effectively act as free electrons, increasing the number density and reducing ρ

Ohm's Law

Qualitatively the distinction between a device that obeys **Ohm's law**, like a resistor, and one that does not, like a diode, may be seen in the figure below:





Ohm's Law is the statement that the current through a device is proportional to the potential difference applied to the device.

A conducting device obeys Ohm's Law when the resistance of the device is independent of the magnitude and polarity of the applied potential difference.

A conducting device obeys Ohm's Law when the resistivity of the material is independent of the magnitude and polarity of the applied electric field.

A material may obey Ohm's law within a finite range of applied voltages / electric fields, but deviate from this linear response if the applied voltages / electric fields are too great.

From Newton's 2nd Law a conduction electron accelerates:

$$a = F/m = eE/m$$

Kinematics gives:

$$v_d = a\tau = eE\tau/m$$

Where τ is the time in between **e^- - atom** collisions.

Since $v_d = \dfrac{I}{neA} = \dfrac{J}{ne}$ ➔ $E = \dfrac{m}{e^2 n\tau} J$ **Or** $\rho = \dfrac{m}{e^2 n\tau}$

Constancy of τ ➔ constancy of ρ and a linear relationship between field and current density.

This is Ohm's Law from a microscopic perspective $E = \rho J$

The analogous macroscopic device statement of Ohm's Law is $V = IR$.

Power in Electric Circuits

Electrons moving from the negative battery terminal through a circuit to the positive terminal undergo an electric potential increase, but an electric potential energy decrease:

$$dU = dq(V_f - V_i)$$

Here dq is an infinitesimal amount of negative charge.

The battery potential energy is either transferred into the circuit as heat in the form of **resistive losses** or as energy supplied to a load attached to the circuit, or both.

Resistance to electron flow through a device or resistor results from collisions between the conduction electrons and resistor molecules.

These collisions generate **thermal energy** that increases the resistor temperature.

The rate at which this transfer takes place is the **power** delivered to the circuit:

$$P = \frac{dU}{dt} = \frac{dq}{dt}(V_f - V_i) = IV$$

Using Ohm's Law, this can also be written in terms of the resistance as:

$$P = IV = I^2 R \qquad Or \qquad P = IV = V^2 / R$$