

Electric Potential Energy and Electric Potential

A scalar field, involving magnitudes only, is often easier to work with when compared to a vector field. For electric fields not having to begin with vector issues would be nice. To arrange this a scalar field is defined from which the electric field may be derived.

Recall from gravitation that the gravitational force is a conservative force, which implies the following attributes for the gravitational force:

- I. A potential energy term may be deduced from which energy associated with the charges spatial configuration (**gravitational charge = mass**) can be evaluated.
- II. The work done by the field as a charge is moved in a closed loop within that field is zero, and the work done on the charge in moving along an arbitrary path depends only on the beginning and ending points.

- III. Path independence is $\oint \vec{F} \cdot d\vec{l} = 0$ A line integral giving the work done by the field during 'charge' motion in a closed loop.

A similar $\frac{1}{r^2}$ dependence in the electrostatic force means that it is also conservative and the above results hold for the electrostatic force.

Given the force, a potential energy term derives from $\vec{F} = -\nabla U$

When $\vec{F}$ is radially dependent only, then $q\vec{E} = -\frac{\partial U}{\partial r} \hat{r}$

Electric field is therefore available via the scalar electric potential energy U .

Integrating $q\vec{E} = -\partial U / \partial r$, and dividing by the test charge q defines the

electric potential difference

$$\Delta V = -\int_i^f \vec{E} \cdot d\vec{r}$$

The electric potential difference is measured in volts, (A. Volta 1745 - 1827)

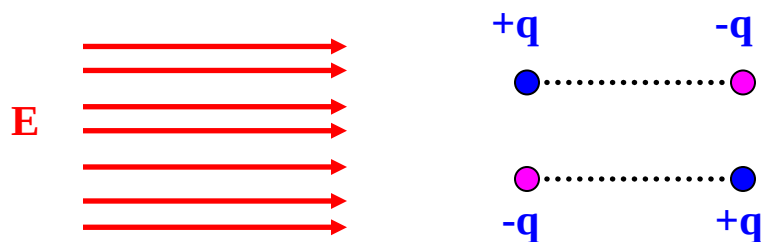
What is electric potential energy?

Potential energy is the energy associated with a particular physical configuration.

With the gravitational force, if mass is elevated to a height **above ground level** then this is a configuration with more potential energy.

The analogous electrostatic case must somehow involve electric charges being '**elevated**' .

Consider two electric dipole configurations in an external electric field:



Having agreed that given the **E** was established by a line of **+** charges, then the **1st** configuration (**+q 'closest' to + charges of E**) has more 'potential energy' as follows:

When a mass is released from some height, a net force from gravity on the object accelerates the mass transferring its potential energy into kinetic energy.

Fixing the center of the dipole as an axis of rotation, any small perturbation away from horizontal on the top dipole leads to a transition into the more stable **2nd** configuration.

The **2nd** configuration corresponds to the minimum electric potential energy (mass on the ground) and the **1st** configuration is at higher potential energy.

Electric potential energy is energy of a charged object in an external electric field.

If we 'release' the charge, then work done by the electric field on the charge:

$$W_{\text{Done_by_the_field}} = -\Delta U$$

For the dipole, as the **1st** configuration moves to 'ground' $\Delta U = -2PE$ and the field does positive work as the dipole 'falls' into the **2nd** configuration.

Electric Potential

From
$$\Delta V = \frac{1}{q} \int_i^f \partial U = - \int_i^f \vec{E} \cdot \vec{ds}$$
 the charge on the test object is

removed from the problem when electric potential is defined.

That is, V represents the potential energy per unit charge due the electric field:

The electric potential is a property of an existing electric field, regardless of whether or not a charged object is placed in the field.

Dimensionally, V has SI units $\frac{J}{C}$ Or $\frac{N}{C}m$

1Volt = 1Joule/Coulomb and electric fields have units of **Volts/Meter**.

The connection to a battery or electric outlets as sources of electricity (voltage) is that the voltage supplied is the result of a charge separation that can establish an electric field which may be used to produce 'currents', i.e., plug-it-in.

Once we have the electric potential in a system containing electric charges (discrete or

continuous), we can use
$$\Delta V = -\int_i^f \vec{E} \cdot \vec{ds}$$
 to obtain the electric field.

The field-mapping laboratory is an example where the field was deduced using a set of equipotential surfaces. Electric field direction is perpendicular to equipotentials.

Electric Potential of a Point Charge

As in the case of gravity we take electric potential energy to be zero at $\mathbf{r} = \infty$:

Then the electric potential at a location **R** from the point charge may be found:

$$\Delta V = V_f - V_i = V_R - 0 = -\int_{\infty}^R \vec{E} \cdot \vec{ds} = -\int_{\infty}^R E(dr)$$

$$V_R = -\int_{\infty}^R \frac{kQ}{r^2} dr = \frac{kQ}{r} \Big|_{\infty}^R = \frac{Q}{4\pi\epsilon_0 R}$$

For a **+Q** establishing the electric field, V is a positive scalar.

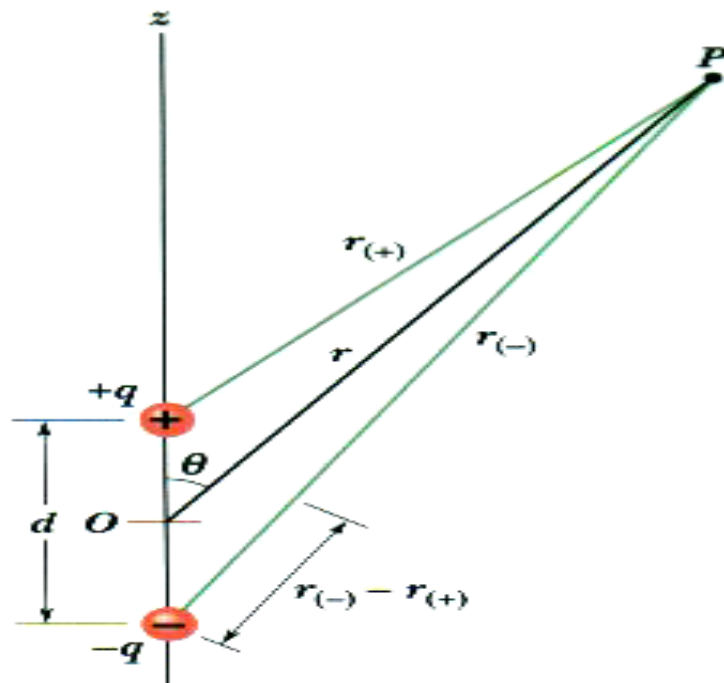
For a **-Q** establishing the electric field, V is a negative scalar.

For a system of discrete charges:

$$V = \sum_{i=1}^n V_i = k \sum_{i=1}^n \frac{q_i}{r_i}$$

Once a method for calculating electric field from electric potential is known, calculating the electric potential and deriving an electric field is much easier than doing the vector algebra problem associated with a superposition of electric fields from each charge.

Electric Potential from a Dipole



$$V = \sum_{i=1}^2 V_i = k \left\{ \frac{q}{r_+} - \frac{q}{r_-} \right\} = kq \left\{ \frac{r_- - r_+}{r_- r_+} \right\}$$

$$r_- - r_+ = \sqrt{r^2 + \left(\frac{d}{2}\right)^2 - 2r \frac{d}{2} \cos(\pi - \theta)} - \sqrt{r^2 + \left(\frac{d}{2}\right)^2 - 2r \frac{d}{2} \cos(\theta)}$$

For $r \gg d$ the binomial expansion of these two terms gives: $(1+x)^n \sim 1+nx$

$$r_- - r_+ = r \left(1 - \frac{1}{2} \frac{d}{r} \cos(\pi - \theta)\right) - \left[r \left(1 - \frac{1}{2} \frac{d}{r} \cos(\theta)\right)\right] = d \cos(\theta)$$

And the product $r \cdot r$ is just $r^2 \rightarrow$ The resulting potential from the dipole is

$$V = \frac{kqd \cos(\theta)}{r^2} = \frac{P \cos(\theta)}{4\pi\epsilon_0 r^2}$$

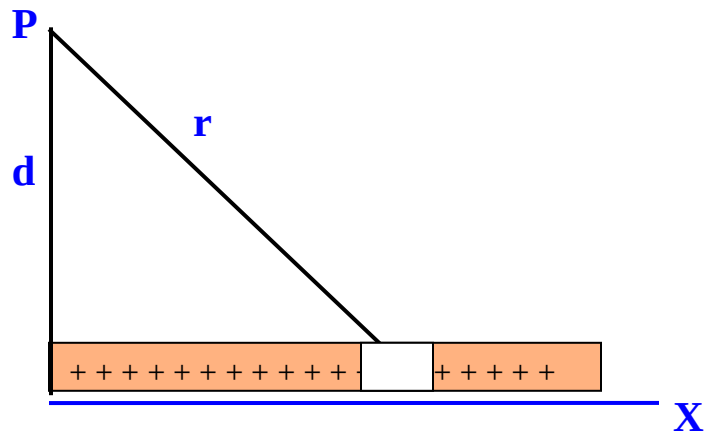
Potential of Continuous Charge Distributions

For elements of differential charge within the distribution, integrate to obtain the electric potential as follows:

$$dV = \frac{k dq}{r}$$

$$V = \int \frac{k dq}{r}$$

Electric potential produced by a finite length of charge with a charge per unit length $\lambda = q/L$.



$$dV = \frac{k dQ}{r} = k \frac{\lambda dx}{r} = k \frac{\lambda dx}{\sqrt{x^2 + d^2}}$$

$$V = \int_0^L k \lambda \frac{dx}{\sqrt{x^2 + d^2}}$$

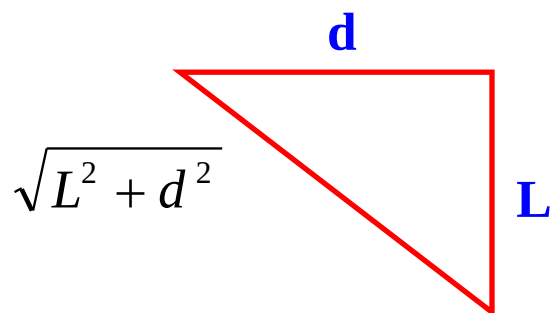
From the figure, $x = d \cdot \tan(\theta) \rightarrow dx = d \cdot (\sec^2 \theta) d\theta$

The radical has $x^2 + d^2 = d^2 (1 + \tan^2(\theta)) = d^2 (\sec^2(\theta))$

$$V = \int_0^{\tan^{-1} \frac{L}{d}} k\lambda \sec(\theta) d\theta$$

$$V = k\lambda \left\{ \ln \left| \sec(\theta) + \tan(\theta) \right| \right\}_0^{\tan^{-1} \frac{L}{d}}$$

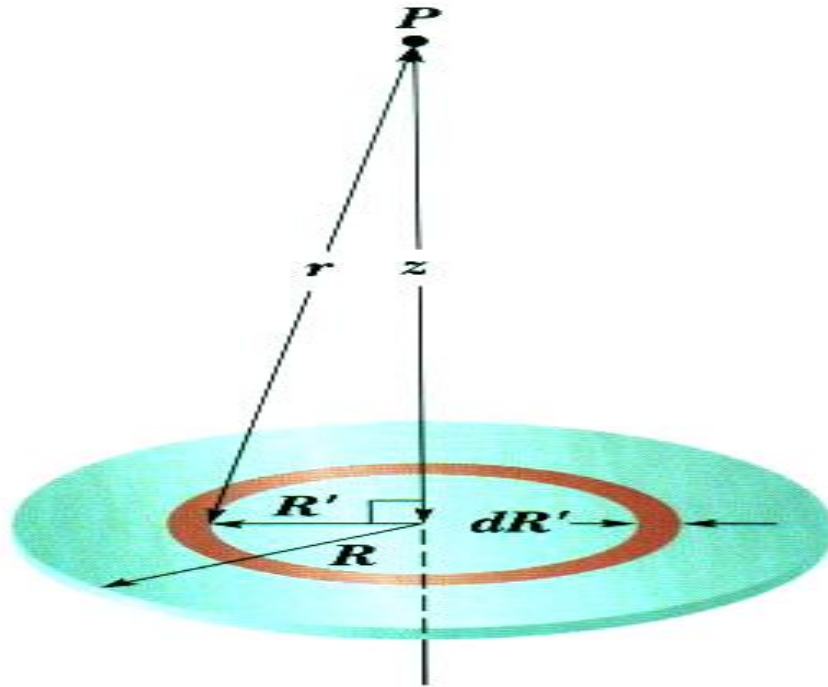
$$V = k\lambda \left\{ \ln \left| \sec\left(\tan^{-1} \frac{L}{d}\right) + \frac{L}{d} \right| \right\}$$



$$V = k\lambda \left\{ \ln \left| \frac{\sqrt{L^2 + d^2}}{d} + \frac{L}{d} \right| \right\}$$

On-Axis Electric Potential from a Charged Disk

$$V = \int \frac{k dq}{r} \qquad dq = \sigma dA = \sigma 2\pi R' dR'$$



$$V = \int_0^R \frac{k\sigma 2\pi R' dR'}{\sqrt{R'^2 + z^2}}$$

Let $u = R'^2 + z^2$ $du = 2R' dR'$

$$V = k\sigma\pi \int_{z^2}^{R^2+z^2} \frac{du}{\sqrt{u}}$$

$$V = \frac{\sigma}{2\epsilon_0} \left\{ \sqrt{R^2 + z^2} - z \right\}$$

Note the asymptotic limits for the variable Z.

Calculating Electric Fields from Electric Potentials

Starting from the definition of electric potential we can differentiate on both sides:

$\Delta V = - \int_i^f \vec{E} \cdot d\vec{s}$	$\vec{E} = -\nabla V$
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Recall the force is deducible from the potential energy as $\vec{F} = -\nabla U$

The gradient of the electric potential gives the electric field:

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial x} \hat{i} - \frac{\partial V}{\partial y} \hat{j} - \frac{\partial V}{\partial z} \hat{k}$$

$$V = \frac{\sigma}{2\epsilon_0} \left\{ \sqrt{R^2 + z^2} - z \right\}$$

Charged Disk Potential Is

$$\vec{E} = -\frac{\sigma}{2\epsilon_0} \frac{\partial}{\partial z} \left\{ \sqrt{R^2 + z^2} - z \right\} \hat{k} = -\frac{\sigma}{2\epsilon_0} \left\{ \frac{z}{\sqrt{R^2 + z^2}} - 1 \right\} \hat{k}$$

Electric Potential Energy of a System of Point Charges

Given one charge produces an electric potential

$$V_1 = \frac{kq_1}{r}$$

Then the work done by an external agent in bringing another charge q_2 into this region from infinity is minus the work done by the existing electric field or $+\Delta U$

$$+\Delta U = U_f - U_i = q_2 V_1 - 0$$

At this point the new electric potential is

$$V_2 = \frac{kq_1}{r_1} + \frac{kq_2}{r_2}$$

Now the work done by an external agent in bringing another charge q_3 into this region from infinity is minus the work done by the new electric field or $+\Delta U$

$$+\Delta U = U_f - U_i = q_3 V_2 - 0$$

The new electric potential is then

$$V_3 = \frac{kq_1}{r_1} + \frac{kq_2}{r_2} + \frac{kq_3}{r_3}$$

Subsequent iterations lead to the conclusion that the assembly of a group of **n** point electric charges in a finite region of space requires an amount of work done $+ \Delta U$

$$\Delta U = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{kq_i q_j}{r_{ij}} \quad i \neq j$$

The electric potential energy of a system of point charges is equal to the work needed to assemble the configuration, all charges initially an infinite distance apart.

Electric Potential of a Charged Isolated Conductor

From the definition of electric potential and the condition of electrostatics, general properties for a charged isolated conductors may be deduced:

$$\Delta V = - \int_i^f \vec{E} \cdot d\vec{s}$$

Zero current requires $\vec{E} = 0$ inside the conductor since free electrons otherwise flow.

Gauss' Law then implies that all excess charge on the conductor resides on its surface.

$\vec{E} = 0 \rightarrow$ the surface of a conductor is an equipotential or $\Delta V = 0$

$\Delta V = 0$ Implies $\vec{E} \cdot d\vec{s} = 0 \rightarrow$ electric field is perpendicular at the conductor surface.

Non-uniform conductors with sharp edges have higher $\vec{E}$ in these areas.

In a static external field, charges on a conductor redistribute to maintain

$\vec{E} = 0$ inside:

