

Electric Field Flux & Gauss' Law

Given a charge distribution with an associated electric field, an often-convenient method of determining the field is by examination of electric field lines near the distribution. This approach is most effective if a high degree of geometric symmetry exist within the charge distribution.

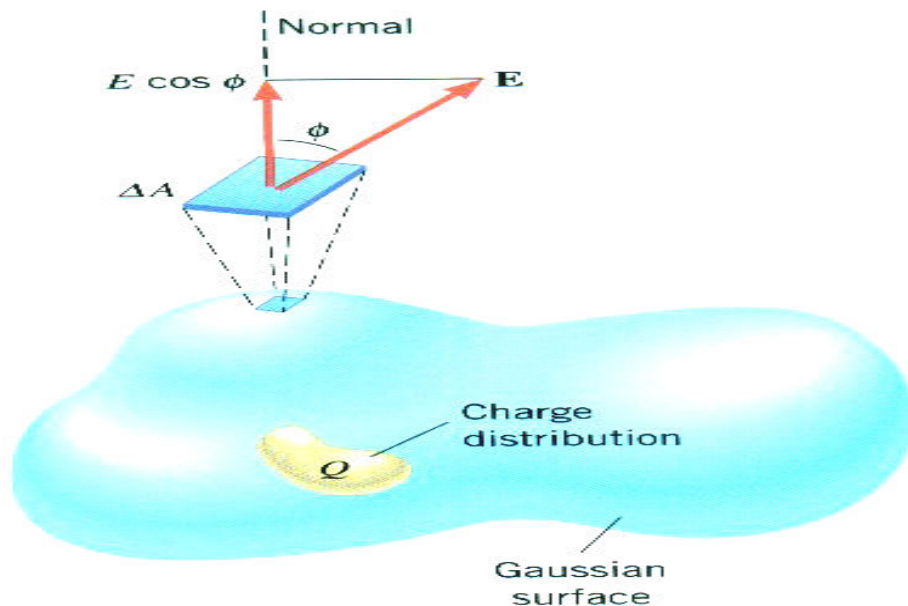
If we choose a hypothetical closed surface ("Gaussian Surface") in which to **enclose** the symmetric charge distribution, then by evaluating the flux of the electric field lines either into or out of the enclosed region, the associated electric field is found using Gauss' Law

- 1) The flux of any set of field lines across an arbitrary surface area corresponds to the number of field lines passing through this surface at normal incidence.

$$\Phi = \int \vec{E} \cdot \hat{n} dA$$

The dot product assures normality and the integration sums over all field contributions.

Field lines 'entering' {crossing A anti-parallel to $\hat{n}$ } produce 'negative' flux and field lines 'leaving' {crossing A parallel to $\hat{n}$ } correspond to 'positive' flux.



Integration over a **closed surface**, preferably one with symmetry such that integration is tractable, gives an integrated flux that is directly proportional to the charge enclosed by the surface:

The net field passing through a surface area element dA that is also perpendicular to the normal of the surface area element is $E_{\perp} = E \cos(\phi)$

The differential flux element is $d\Phi = \vec{E} \cdot \hat{n} dA$

Integrating this over a closed surface gives the net electric field flux into or out of the region enclosed as:

$$\Phi = \oint \vec{E} \cdot \hat{n} dA$$

Evaluating this quantity for a **point charge** enclosed in a **gaussian sphere**, we find that since the electric field is uniform over the sphere surface the integration is simplified:

$$\Phi = \oint \vec{E} \cdot \hat{n} dA = E \oint dA = E 4\pi r^2$$

From which

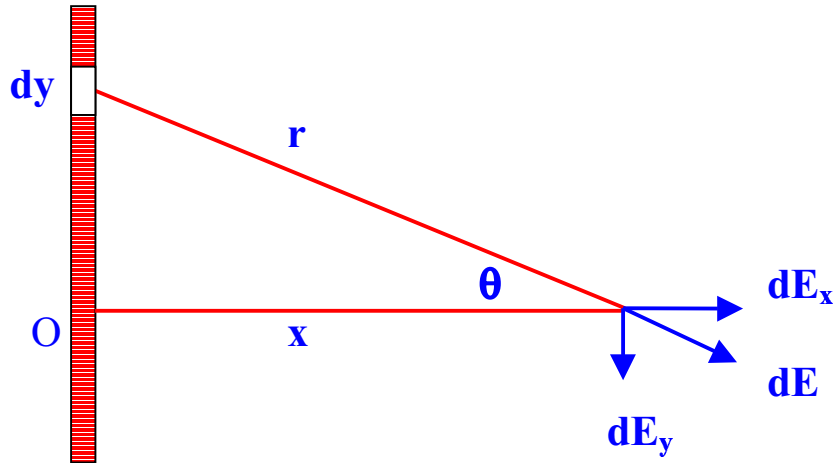
$$E = \frac{q}{4\epsilon_0 \pi r^2}$$

And therefore,

$$\Phi = \oint \vec{E} \cdot \hat{n} dA = \frac{q_{\text{enclosed}}}{\epsilon_0}$$

Example

Recall the electric field produced by an infinite length of charge with linear charge density λ was:



$$dE = \frac{k dQ}{r^2} = k \frac{\lambda dy}{r^2} = k \frac{\lambda dy}{x^2 + y^2}$$

$$E = E_x = \int dE \cdot \cos(\theta) = k\lambda \int \cos(\theta) \frac{dy}{x^2 + y^2}$$

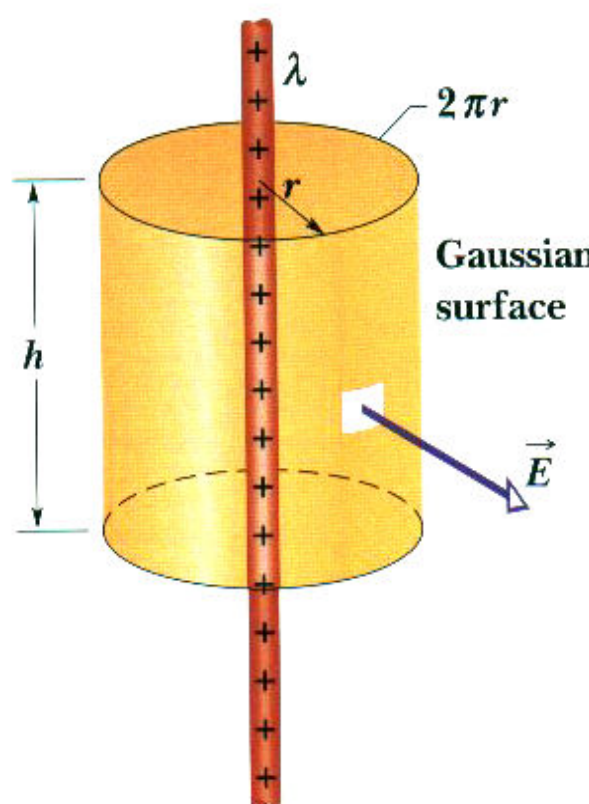
From the figure $y = x \cdot \tan(\theta) \Rightarrow dy = x \cdot (\sec^2 \theta) d\theta$

The denominator is $x^2 + y^2 = x^2 (1 + \tan^2(\theta)) = x^2 (\sec^2(\theta))$

$$E = k\lambda \int \cos(\theta) \frac{dy}{x^2 + y^2} = \frac{k\lambda}{x} \int_{-\pi/2}^{\pi/2} \cos(\theta) d\theta = \frac{2k\lambda}{x}$$

Using Gauss' Law for an infinite line of charge is considerably easier as follows:

$$\oint \vec{E} \cdot \hat{n} dA = q_{\text{enclosed}} / \epsilon_0$$

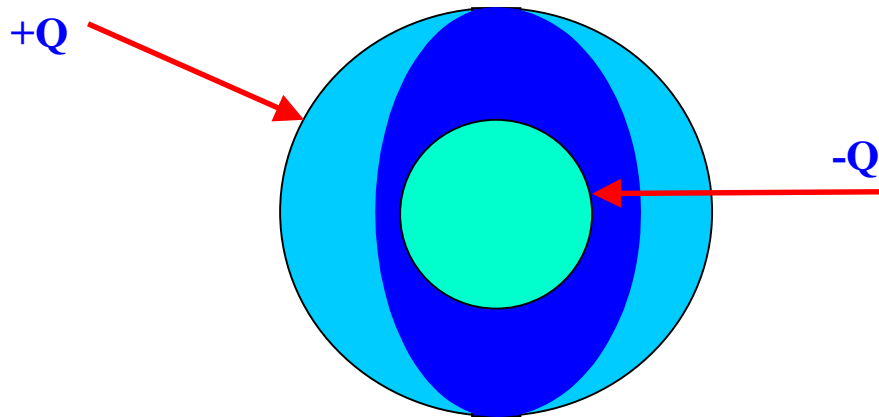
 The diagram shows a vertical red line representing an infinite line of charge with a linear charge density λ, indicated by '+' signs. A yellow cylindrical Gaussian surface of radius r and height h is centered on the line. The top circular face of the cylinder is labeled with the circumference $2\pi r$. The label 'Gaussian surface' points to the cylinder. A blue arrow labeled $\vec{E}$ points radially outward from the line, representing the electric field.	Since the field is uniform on the cylindrical surface and everywhere normal to the surface area, the electric field may be factored from the integrand: $E \oint dA = q_{\text{enclosed}} / \epsilon_0$ Leading to: $E 2\pi r h = \lambda h / \epsilon_0$ $E = 2 k \lambda / r$
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Example: problem with spherical symmetry:

Consider the two concentric conducting shells with charge as shown:

The electric field outside the outer shell is zero by Gauss' Law.

For the location inside the $+Q$ shell, but outside the $-Q$ shell, Gauss' Law has



$$\vec{E} = -Q / 4\pi\epsilon_0 r^2 \quad \text{radially_inward}$$

Inside the $-Q$ shell, the field is also zero by Gauss' Law.

Shell Theorems

A conducting Shell of uniform σ attracts q outside as if all charge were located at the shell center.

A conducting Shell of uniform σ has no effect on charges within the shell.

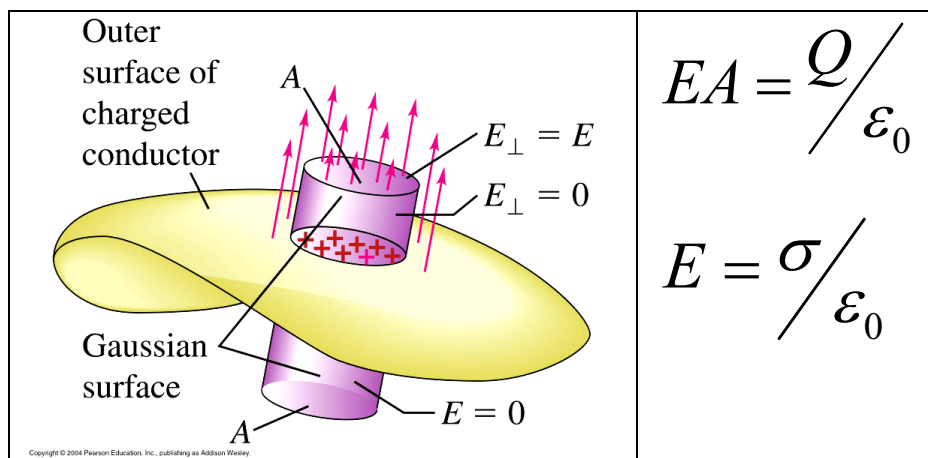
Charged Isolated Conductors

Charged → Excess of charge was added or removed from the object. (Usually electrons)

Isolated → No other charge sources or sinks in the region. (Not grounded in particular)

Conductors → The objects electrons are free to move when an electric field is applied.

- I. $\vec{E}$ Is identically zero inside a conductor since otherwise currents would flow continuously which is not the purview of electrostatics.
- II. Excess of charge must reside **on** the conductor surface. Since the electric field is zero in the conductor, work Gauss' Law backwards to conclude that there must be **zero charge inside** the conductor.
- III. Non-spherically shaped conductors have non-uniform charge densities on their surface and electric field values are highest near corners and edges.
- IV. $\vec{E}$ At the outside surface of a conductor is perpendicular to the conductor surface. Any transverse field would establish currents within the conductor.
- V. $\vec{E}$ at the surface of a charged conductor is $\frac{\sigma}{\epsilon_0}$ by Gauss' Law:



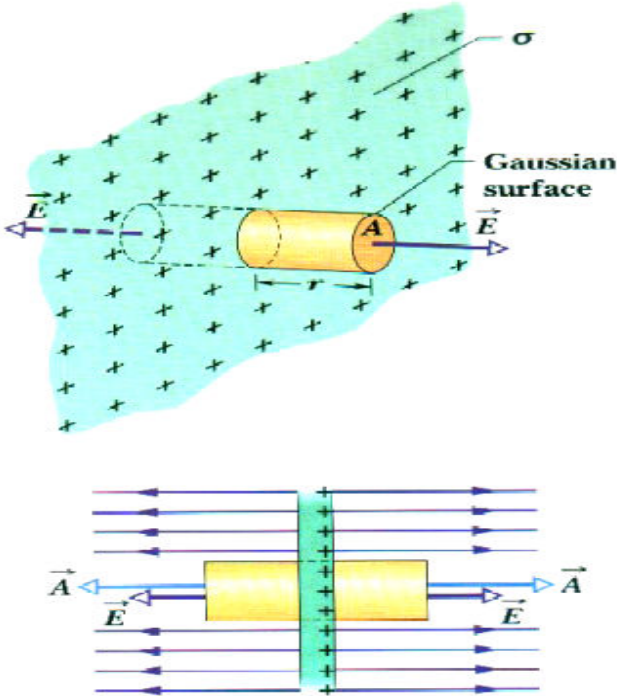
When a conducting sheet is carrying an excess of electric charge, the field in the conductor must be zero.

How is the result changed if the sheet is an insulator or dielectric material?

Conducting sheet of material VS. Non-conducting sheet of material.

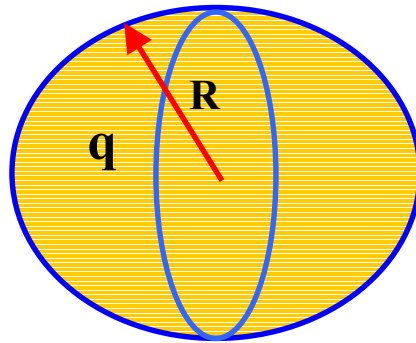
No longer constrained to have zero electric field in the insulating material, Gauss'

Law gives the $\vec{E}$ at the non-conducting surfaces:

	For a <u>non-conducting</u> sheet of material with an excess of electric charge, Gauss' Law is: $\oint \vec{E} \cdot \hat{n} dA = q_{enclosed} / \epsilon_0$ $EA + EA = q / \epsilon_0$ $E = \sigma / 2 \epsilon_0$
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Non-conducting Sphere of Uniform Charge Density

$$\rho = \frac{q}{\frac{4}{3}\pi R^3} = \text{const}$$



For radii less than **R**, Gauss' Law gives

$$\oint \vec{E} \cdot \hat{n} dA = q_{\text{enclosed}} / \epsilon_0 = \frac{1}{\epsilon_0} * \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} q = \frac{qr^3}{\epsilon_0 R^3}$$

The electric field at points within the sphere is then:

$$\vec{E} = \frac{qr}{4\pi\epsilon_0 R^3} \quad \text{radially directed}.$$

For observation points of radius greater than R,

$$\oint \vec{E} \cdot \hat{n} dA = q_{\text{enclosed}} / \epsilon_0 = \frac{1}{\epsilon_0} * \frac{4}{3}\pi R^3 \rho$$

And the electric field:

$$\vec{E} = \frac{q_{\text{enclosed}}}{4\pi\epsilon_0 r^2} \quad \text{radial}$$

$$\vec{E} = \frac{\rho R^3}{3\epsilon_0 r^2}$$