Inductors and Inductance

Circuit elements that produce magnetic fields and have **inductance** have application in stabilizing currents, oscillators and power.

An **ideal solenoid is the inductor prototype**. Inductance is defined:

$$L \equiv \frac{N\Phi}{i} \qquad N = \#\_of\_turns$$

The SI unit of inductance is the **Henry**. **$1H = 1Tm^2A^{-1}$**

$N\Phi$ is the **flux linkage**. **Inductance is: flux linkage per unit current**.

E.g., Ideal Solenoid:

$$B = \mu_0 ni$$

n **here is turns per unit solenoid length.**

$$L \equiv \frac{N\Phi}{i} = \frac{N}{i} * BA = \frac{N\mu_0 niA}{i} = nl\mu_0 nA$$

$$L = \mu_0 n^2 lA = f(Geometry)$$

$$L/l = \mu_0 n^2 A$$

In units of henry/meter.

Note the units of permeability are, therefore, **henry/meter** and recall from capacitance that the units of permitivity are **farad/meter**.

Self Induction

Given an inductor and given a changing current ${di}/{dt}$ within its coils, this changing current produces a changing $\vec{B}$ in the inductor, and therefore a **self-induced emf**.

The induced **current direction** is such that the induced magnetic field produced from this current counters whatever change is occurring in the inductor flux.

$$E = -N\frac{d\Phi}{dt} = -N\frac{d}{dt}(BA) = -NA\frac{d}{dt}\mu_0 ni = -\mu_0 n^2 lA\frac{di}{dt} = -L\frac{di}{dt}$$

$$E = -L\frac{di}{dt}$$

For steady currents, the induced emf is zero.

For an **ideal inductor**, the resistance of the wire material is negligible, and the voltage across the inductor is the self-induced emf.

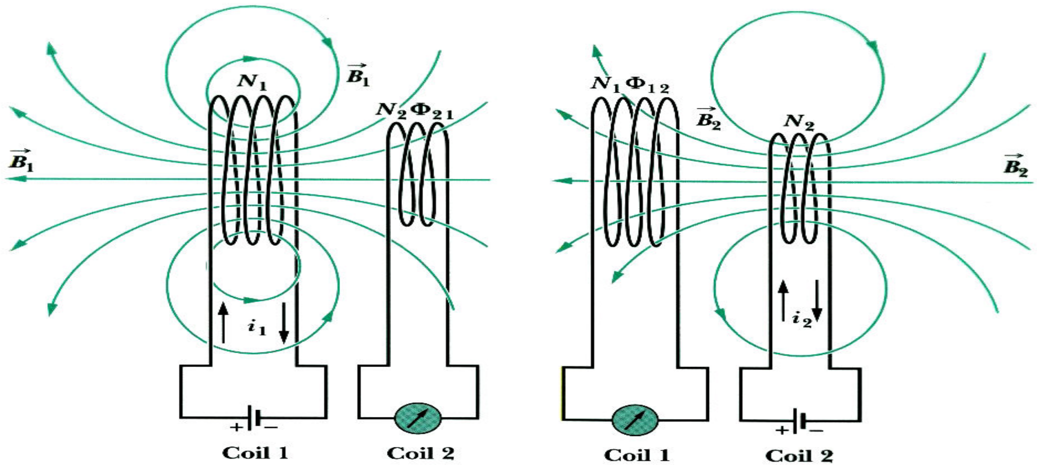
If **current is steady** the inductor behaves as a short with zero voltage drop.

In practice, **real inductors** may be modeled by the series combination of an inductor resistance outside the region of changing magnetic fields plus an ideal inductor.

Mutual Inductance

Consider interactions between two inductors in 'close' proximity. As current in inductor #1 changes in time, this produces a changing magnetic field in inductor #1 and therefore a changing flux in nearby inductor #2. Changing flux in inductor #2 produces an emf in inductor #2. This is **mutual inductance**.

Similarly, any changes in inductor #2 current will induce an emf in inductor #1.



The mutual inductance of inductor 2 with respect to inductor 1 is:

$$M_{21} = \frac{N_2\Phi_{21}}{i_1}$$ As with self-inductance, M is characterized by geometry.

$$M_{21}\frac{di_1}{dt} = N_2\frac{d\Phi_{12}}{dt} = -E_2$$

$$E_2 = -M_{21}\frac{di_1}{dt}$$ **Induced emf in inductor #2 from changing current in #1**

Similarly $$E_1 = -M_{12}\frac{di_2}{dt}$$

Usually a small-unwanted effect in circuitry, mutual inductance is also the way step-up / step-down **transformers** operate.

Magnetic Field Energy Density

The energy stored in an inductor is **magnetic field energy**. $u = {U_B}/{V}$

$$u = \frac{1}{V}\int P dt = \frac{1}{lA}\int i v_L dt = \frac{1}{lA}\int iL\frac{di}{dt}dt = \frac{L}{l}\frac{i^2}{2A}$$

Which, given $L/l = \mu_0 n^2 A$ for a solenoid, this reduces to $u = \frac{1}{2}\mu_0 n^2 i^2$

Using $B = \mu_0 n i$ gives: $u = \frac{B^2}{2\mu_0}$ Energy density of magnetic field.

With the presence of magnetic material permeability μ, then $u = \frac{B^2}{2\mu}$

RL Circuits

In an approach analogous to what was done with charging/discharging in an RC circuit, the current response in an RL circuit may be evaluated starting from Kirchhoff's rule:

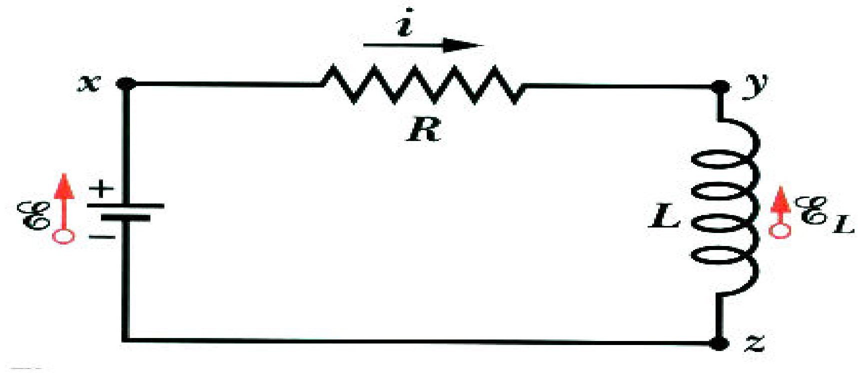
Recall the RC circuit results as follows:

$$E - iR - \frac{Q}{C} = 0 \qquad \frac{dQ}{dt} + \frac{1}{RC}Q - \frac{E}{R} = 0$$

Charging $Q = CE(1 - e^{-t/\tau})$ $\qquad \tau = RC$

Discharging $Q = Q_0 e^{-t/\tau}$

In an **RL circuit**, the inductor initially appears as **open** and eventually **t→ ∞** as a **short**.



$$E - iR - L\frac{di}{dt} = 0 \qquad\qquad \frac{di}{dt} + \frac{R}{L}i - \frac{E}{L} = 0$$

Taking $Q \to i$ & $\frac{1}{RC} \to \frac{R}{L}$

$$i(t) = \frac{E}{R}(1 - e^{-t/\tau}) \qquad\qquad \tau = \frac{L}{R} = Inductive\_Time\_Const.$$

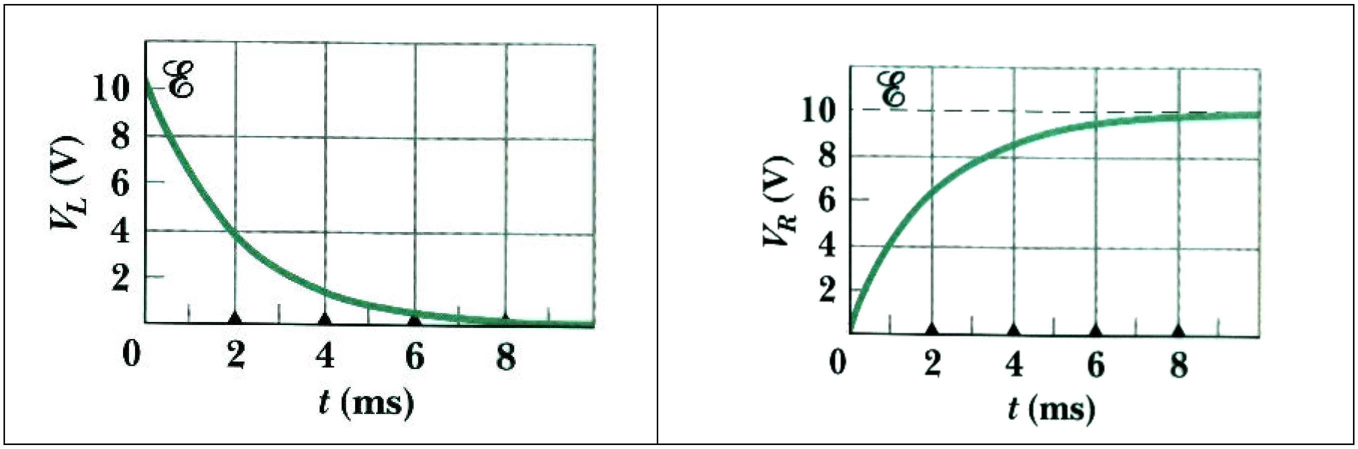
The voltage on the inductor is a maximum to start and exponentially tends to zero:

$$V_L = -L\frac{di}{dt} = \frac{LE}{R\tau}e^{-t/\tau} = Ee^{-t/\tau}$$

Using Kirchhoff's loop rule, the resistor voltage is therefore:

$$V_R = E - V_L = E - Ee^{-t/\tau} = E(1 - e^{-t/\tau})$$

Zero to start and asymptotically tending to E as the inductor appears to short.



Removing the circuit emf, $$i(t) = i_0 e^{-t/\tau} = \frac{E}{R} e^{-t/\tau}$$

Inductor Energy

The RL circuit differential equation is $$E - iR - L\frac{di}{dt} = 0$$

Multiplying this equation by current $$iE - i^2R - Li\frac{di}{dt} = 0$$

These energy terms from left to right are:

1) Rate at which energy is delivered to the circuit by the power supply
2) Rate at which thermal energy is produced by resistor heating
3) Rate at which energy is stored in the inductor magnetic field.

The last of these is: $$\frac{dU_B}{dt} = Li\frac{di}{dt} \quad \Rightarrow \quad U_B = \frac{1}{2}Li^2$$

LC Oscillators

Directly analogous to the **mass-spring oscillating mechanical system**, the electronic counterpart is that of an **inductor-capacitor oscillating circuit**.

Oscillations are **electromagnetic oscillations** of capacitor electric fields and inductor magnetic fields.

The circuit has all the properties of an oscillating system including a resonance driving frequency, in this case $\omega = 1/\sqrt{LC}$ vs. $\omega = \sqrt{k/m}$ for the mechanical system.

A correspondence of $L \rightarrow m \quad and \quad C \rightarrow k^{-1}$ is made.

Solutions to the differential equations describing the capacitor charge $q(t)$ may be extracted from the mechanical system solutions by making the appropriate variable substitutions into those results.

Simple Harmonic Oscillation

A capacitor charge that is **periodic** or repeats in regular time intervals, and is a sinusoidal or co-sinusoidal function of time is referred to as **simple harmonic** in time.

$$q(t) = QCos(\omega t + \phi)$$

Q is the **amplitude** and is the maximum +/- capacitor charge.

ω is the **angular frequency** of the oscillator and related to the frequency by $\omega = 2\pi f$

ϕ is a **phase factor or phase angle** in units of radians.

f is the **frequency** or number of oscillations per second. Units of **f** are Hertz, Hz.

LC Circuit $\Leftrightarrow$ Mass-Spring Analogy

Newton's 2nd Law for the mass-spring oscillator is:

$$m\frac{d^2x}{dt^2} = -kx \qquad\qquad m\frac{d^2x}{dt^2} + kx = 0$$ 2nd order diff. eqn.

Try $x(t) = x_m Cos(\omega t)$ as a solution

$$\dot{x}(t) = -\omega x_m Sin(\omega t)$$

$$\ddot{x}(t) = -\omega^2 x_m Cos(\omega t)$$

Substituting: $$-m\omega^2 x_m Cos(\omega t) + kx_m Cos(\omega t) = 0$$

This equation is true iff $\omega = \sqrt{k/m}$

From the circuit energies, and **conservation of energy without resistive damping**, electromagnetic oscillations take place as follows:

$$E = \frac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C} = Total\_Energy = const.$$

$$\dot{E} = 0 = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt}$$

$$\frac{d^2q}{dt^2}+\frac{1}{LC}q=0$$ Compared To $$\frac{d^2x}{dt^2}+\frac{k}{m}x=0$$

Using $L\to m$ & $C\to k^{-1}$, then $q\to x$ is also appropriate.

For the circuit oscillator, 'kinetic' and 'potential' energies are the energies within the inductor and capacitor respectively. The capacitor is spring like in its electric potential energy, inductance mass like, and current a 'velocity' term.

$$q(t)=QCos(\omega t+\phi) \qquad \omega=1/\sqrt{LC}$$

$$i(t)=\frac{dq}{dt}=-\omega QSin(\omega t+\phi)=-ISin(\omega t+\phi)$$

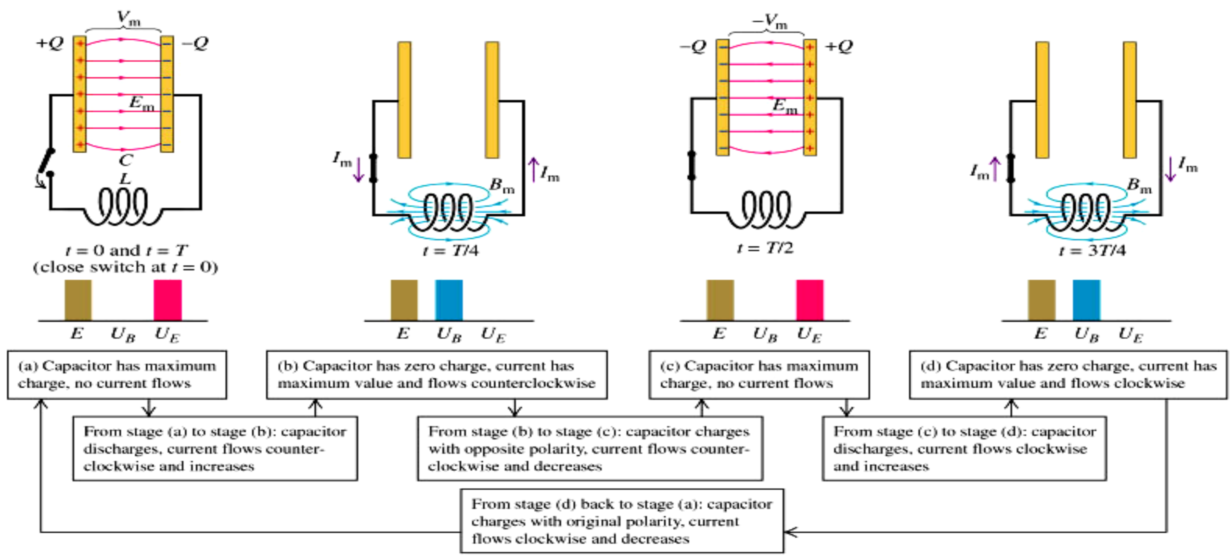
$$I=\omega Q=current\_amplitude$$

Let $\phi=0$ and 'extend the mass', i.e., charge the capacitor to its full charge:

$$q(t=0)=Q \qquad q(t)=QCos(\omega t)$$

$$i(t)=-ISin(\omega t) \qquad i(t=0)=0$$

The system oscillates:



Initially:

$$U_E(t) = \frac{1}{2C}q^2 = \frac{1}{2C}[QCos(\omega t)]^2 = Q^2/2C$$

$$U_B(t) = \frac{1}{2}Li^2 = \frac{1}{2}L*[-ISin(\omega t)]^2 = 0$$

At $t = T/4 = \pi/2\omega$

$$U_E(t) = \frac{1}{2C}q^2 = \frac{1}{2C}[QCos(\omega t)]^2 = 0$$

$$U_B(t) = \frac{1}{2}Li^2 = \frac{1}{2}L*[-ISin(\omega t)]^2 = \frac{1}{2}LI^2$$

The system evolves periodically transferring energy between the capacitor electric field and the inductor magnetic field as shown.

Both 'kinetic' and 'potential' energy peaks occur twice over the course of 1 period **[T].**

Since $\omega = 1/\sqrt{LC}$ the total energy at all times is:

$$E = U_B(t) + U_E(t) = \frac{1}{2C}Q^2\{Sin^2(\omega t+\phi) + Cos^2(\omega t+\phi)\} = \frac{1}{2C}Q^2$$

The electromagnetic energy is a constant provided **resistive damping** is absent.

RLC Circuits, Damped Harmonic Oscillations

Most non-driven oscillating systems will come to rest after a finite amount of time due to dissipative losses. In the RLC circuit, the resistor damps the LC oscillations causing exponentially decaying oscillation amplitude.

$$\dot{E} = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = -i^2R$$

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0$$

Letting $\mathbf{2\gamma = R/L}$ and with $\boldsymbol{\omega_0^2 = 1/LC}$

$$\ddot{q} + 2\gamma\dot{q} + \omega_0{}^2 q = 0$$ As a solution try: $$q(t) = Ae^{Pt}$$

$$P^2 Ae^{Pt} + 2\gamma PAe^{Pt} + \omega_0{}^2 Ae^{Pt} = 0$$

$$P^2 + 2\gamma P + \omega_0{}^2 = 0$$

$$P = -\gamma \pm \sqrt{\gamma^2 - \omega_0{}^2}$$

$$q(t) = Ae^{(-\gamma + \sqrt{\gamma^2 - \omega_0{}^2})*t}$$

Or

$$q(t) = Be^{(-\gamma - \sqrt{\gamma^2 - \omega_0{}^2})*t}$$

The general solution is a linear combination of the two possibilities:

$$q(t) = e^{-\gamma t}\{Ae^{(\sqrt{\gamma^2 - \omega_0{}^2})*t} + Be^{-(\sqrt{\gamma^2 - \omega_0{}^2})*t}\}$$

Three cases exist depending on whether $\gamma^2 < \omega_0{}^2$, $\gamma^2 > \omega_0{}^2$, **or** $\gamma^2 = \omega_0{}^2$

Case 1 Underdamped Solution:

$\gamma^2 < \omega_0^2 \rightarrow$ $R < 2\sqrt{\frac{L}{C}}$

$$q(t) = e^{-\gamma t}\{Ae^{i(\sqrt{\omega_0{}^2 - \gamma^2})*t} + Be^{-i(\sqrt{\omega_0{}^2 - \gamma^2})*t}\}$$

$$q(t) = e^{-\gamma t}\{(A+B)Cos(\sqrt{\omega_0{}^2 - \gamma^2})t + (A-B)iSin(\sqrt{\omega_0{}^2 - \gamma^2})t\}$$

$$q(t) = e^{-\gamma t}\{A'Cos(\omega' t) + B'Sin(\omega' t)\}$$

With initial conditions **q(0) = Q and i(0) = 0,**

A' = Q
B'ω' - A'γ = 0
B' = A'γ/ω' = Q {γ/ω'}

$$q(t) = Qe^{-\gamma t}\{Cos(\omega' t) + \frac{\gamma}{\omega'}Sin(\omega' t)\}$$

Wishing to write something like:

$$q(t) = Q\frac{\omega_0}{\omega'}e^{-\gamma t}\{Cos(\omega' t + \phi)\}$$

We find the phase angle ϕ must be: $\phi = -Tan^{-1}\frac{\gamma}{\omega'}$

Proving this requires:

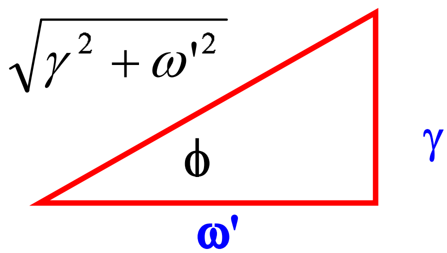
$$\frac{\omega_0}{\omega'}\{Cos(\omega' t+\phi)\} = Cos(\omega' t) + \frac{\gamma}{\omega'} Sin(\omega' t)$$

From trigonometry the LHS is

$$\frac{\omega_0}{\omega'}\{Cos(\omega' t)Cos(\phi) - Sin(\omega' t)Sin(\phi)\}$$

$$Cos(\phi) = Cos\{Tan^{-1}\frac{\gamma}{\omega'}\}$$

The reference triangle is:



$$\omega' = \sqrt{\omega_0{}^2 - \gamma^2} \quad \rightarrow \quad \sqrt{\gamma^2+\omega'^2} = \omega_0$$

$$Cos(\phi) = \omega'/\omega_0 \quad \rightarrow \quad Sin(\phi) = -\gamma/\omega_0$$

From which,

$$\frac{\omega_0}{\omega'}\{Cos(\omega' t+\phi)\} = Cos(\omega' t) + \frac{\gamma}{\omega'} Sin(\omega' t)$$

$$q(t) = Q\frac{\omega_0}{\omega'}e^{-\gamma t}\{Cos(\omega' t + \phi)\} \qquad \phi = -Tan^{-1}\frac{\gamma}{\omega'}$$

The oscillation amplitude is decaying exponentially in time.

Case 2 Critically Damped Solution:

$\gamma^2 = \omega_0{}^2 \rightarrow$ $R = 2\sqrt{\frac{L}{C}}$

In this case ω' is identically zero and we can evaluate the limiting form of:

$$q(t) = Qe^{-\gamma t}\{Cos(\omega' t) + \frac{\gamma}{\omega'}Sin(\omega' t)\}$$

As $\omega' \rightarrow 0$.

$$\lim_{\omega' -> 0}\{Qe^{-\gamma t}[Cos(\omega' t) + \frac{\gamma}{\omega'}Sin(\omega' t)]\}$$

The cosine term goes to 1 in the limit and the sine term is:

$$\lim_{\omega' -> 0}\{\frac{\gamma}{\omega'}Sin(\omega' t)\} = \lim_{\omega' -> 0}\{\gamma t * \frac{Sin(\omega' t)}{\omega' t}\} = \gamma t$$

$$q(t) = Qe^{-\gamma t}\{1 + \gamma t\}$$

The circuit damps to equilibrium as quickly as is possible without oscillation about the equilibrium point.

Case 3; Overdamping:

$\gamma^2 > \omega_0^2 \rightarrow$ $R > 2\sqrt{\frac{L}{C}}$

Starting with $q(t) = Qe^{-\gamma t}\{Cos(\omega' t) + \frac{\gamma}{\omega'} Sin(\omega' t)\}$

$$\omega' = \sqrt{\omega_0^2 - \gamma^2} = i\sqrt{\gamma^2 - \omega_0^2}$$

Since

$$Cos(ix) = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{x} + e^{-x}}{2} = Cosh(x)$$

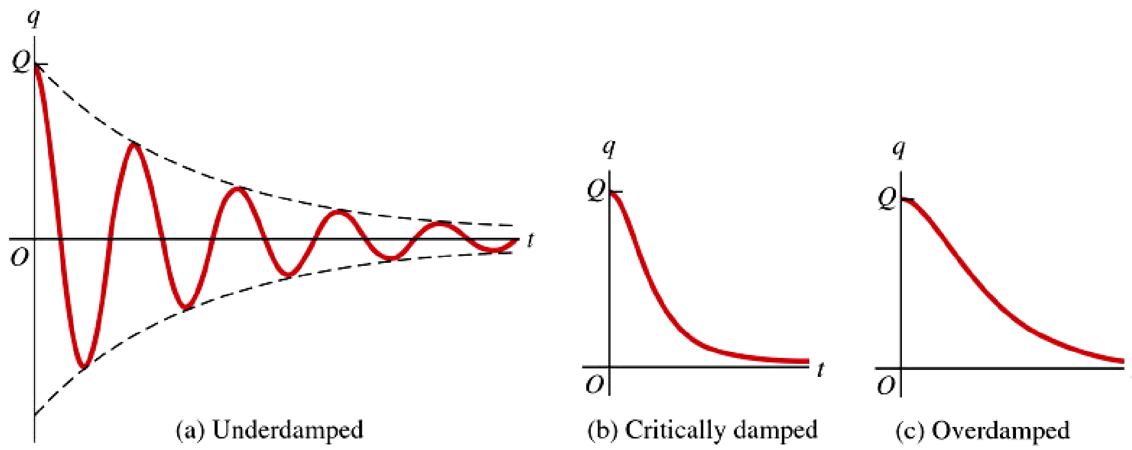
And

$$Sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{x} - e^{-x}}{2i} = iSinh(x)$$

$$q(t) = Qe^{-\gamma t}\{Cosh(\sqrt{\gamma^2 - \omega_0^2}t) + \frac{\gamma i}{\omega'} Sinh(\sqrt{\gamma^2 - \omega_0^2}t)\}$$

$$q(t) = Q\frac{\omega_0}{\omega''}e^{-\gamma t}\{Cosh(\omega'' t + \phi)\}$$

$$\omega'' = \sqrt{\gamma^2 - \omega_0^2}$$



(a) Underdamped circuit (small R)

(b) Critically damped circuit (larger R)

(c) Overdamped circuit (very large R)