Wave Motion

Wave types:

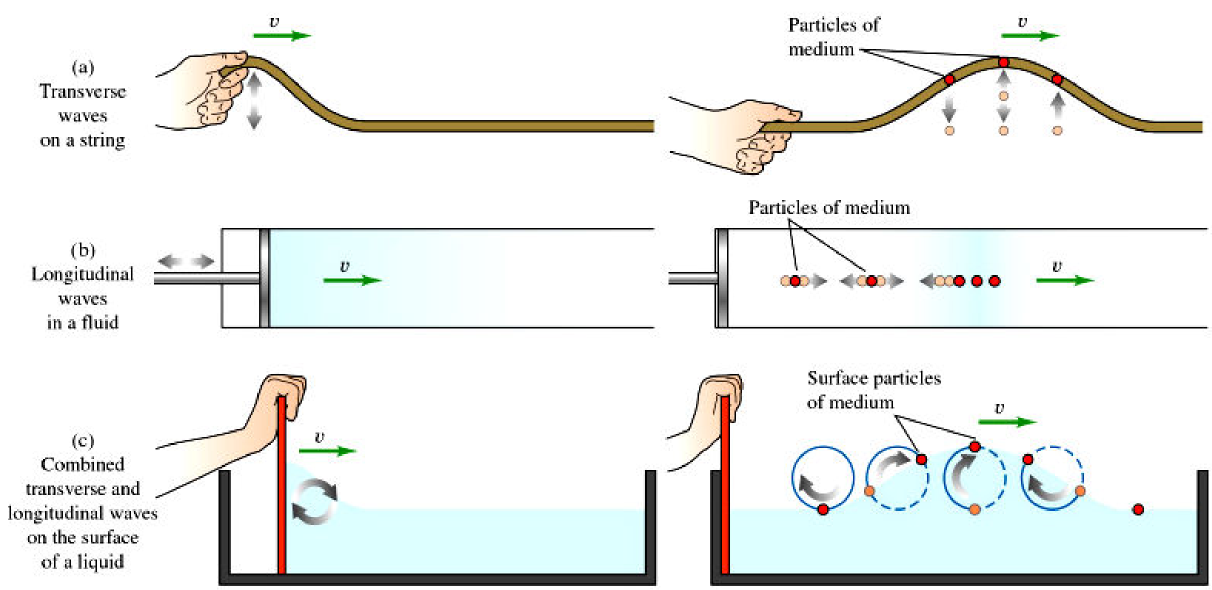
1) **Mechanical Waves** - Described with Newtonian Mechanics

2) **Electromagnetic Waves** - Arrived at Using Maxwell Equations

3) **Matter Waves** - Treated Within the Framework of Quantum Mechanics

Mechanical waves like sound waves, waves along a rope or on a surface, and the waves in a lake consist of disturbances that displace the **medium** through which they travel.

Such waves **transfer energy** from one location to another utilizing the medium.



Mechanical waves are categorized by the direction in which the molecules of the medium are displaced relative to the propagation direction as a **pulse** moves through the medium.

1) **Transverse Waves** - Elements of the medium are displaced perpendicularly to the wave propagation direction.

2) **Longitudinal Waves** - Medium elements are displaced parallel to the wave propagation direction.

3) **Surface Waves** - Displacements are both longitudinal and transverse.

The Scorpion:

Within the solid earth, longitudinal waves propagate from point to point faster than transverse waves. As a hapless Beetle bug moves both types of waves are generated in the ground. A nearby Scorpion will first detect the longitudinal wave indicating the Beetles direction and the time interval between this event and the transverse wave arrival is used by the computationally adept Scorpion to fix a distance to this Beetle snack:

Speeds near the surface are: Transverse V ~ 50 m/s; Longitudinal V ~ 150 m/s

$$\Delta t = \frac{d}{V_T} - \frac{d}{V_L} = \frac{100d}{V_T * V_L} \quad \rightarrow \quad d = (75m/s) * \Delta t$$

Traveling Waves

Consider a non-reflected transverse wave generated by a source oscillating in simple harmonic motion. Let the wave propagate in the +X-direction without energy loss:

Elements of the medium are displaced vertically according to the **wave function**:

$$y(x,t) = ASin(kx - \omega t)$$

A Is the magnitude of maximum displacement w.r.t. equilibrium: the **Amplitude.**

k Is the **Angular Wave Number** in units of radian/meter. $k = 2\pi/\lambda$

ω Is the **Angular Frequency**: $\omega = 2\pi f$.

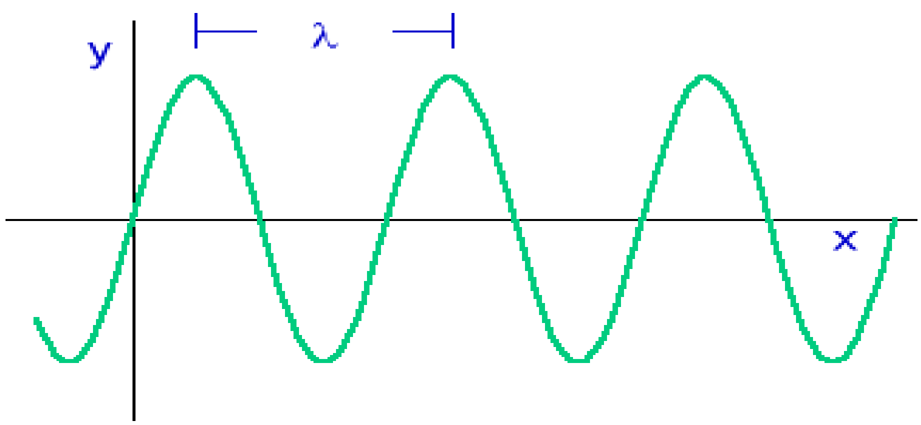
λ Is the **Wavelength** of the wave: Crest to crest distance for example.

$(kx - \omega t)$ Is the **Phase** in radians.

For **non-dispersive waves** where $\omega \neq \omega(t)$, then a constant phase implies that for increasing t, x must also increase, i.e., the wave $y(x,t) = ASin(kx - \omega t)$ represents a wave moving in the **+x** direction.

Other wave parameters are:

1) **Cycle** - One complete medium element oscillation.
2) **Frequency f** - Number of cycles occurring per second.
3) **Period T** - The time interval between repetitions of the wave shape.



At time $t = 0$, $y(x,0) = ASin(kx)$ is the amplitude of the wave at any given **X** coordinate. For a periodic wave, this is also the wave amplitude at one wavelengths distance:

$$y(x,0) = ASin(kx) = y(x+\lambda,0) = ASin(k(x+\lambda))$$

From the sine of a sum of two angles the condition above is true if:

$$k\lambda = 2\pi \qquad\qquad k = 2\pi / \lambda$$

Frequency is related to the wave angular frequency and the period of the wave as:

$$f = \frac{\omega}{2\pi} = \frac{1}{T}$$

Wave Velocities:

The **transverse velocity** of a medium mass element resulting from the passage of a transverse mechanical wave is given by the partial derivative of the position function for that element with respect to the variable t :

$$y(x,t) = ASin(kx - \omega t)$$

$$u(x,t) = \frac{\partial}{\partial t} ASin(kx - \omega t) = -\omega ACos(kx - \omega t)$$

Further, for $(kx - \omega t)$ constant, the **propagation velocity** may be found as:

$$\frac{d}{dt}(kx - \omega t) = 0 \qquad k\frac{dx}{dt} - \omega = 0$$

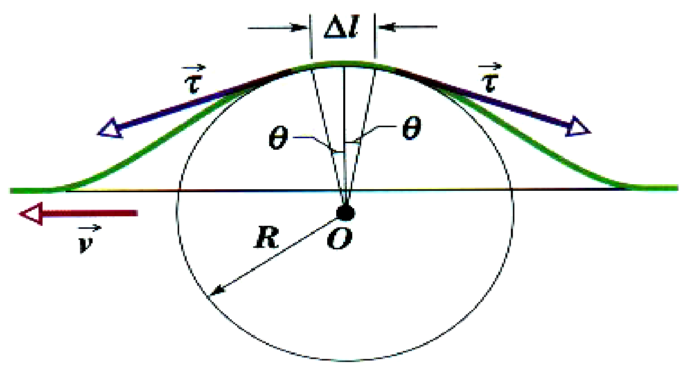
$$\frac{dx}{dt} = \omega / k = f\lambda$$

$$v = f\lambda .$$

Generally, the wave velocity depends on the **medium elasticity** and the **medium inertia**.

Applying Newton's 2nd Law to a transverse wave on a string, we can derive the propagation velocity in terms of the string tension and its linear mass density:

Consider a snapshot of the string:



$$\sum F_y = -\tau \cdot Sin\theta - \tau \cdot Sin\theta = -ma = -\Delta l \mu \frac{v^2}{R}$$

μ = mass / length

$$-2\tau Sin\theta = -\frac{\mu v^2}{R}\Delta l$$

In the limit of small angles theta, $-2\tau\theta = -\frac{\mu v^2}{R}\Delta l$

From geometry, $2\theta * R = \Delta l$

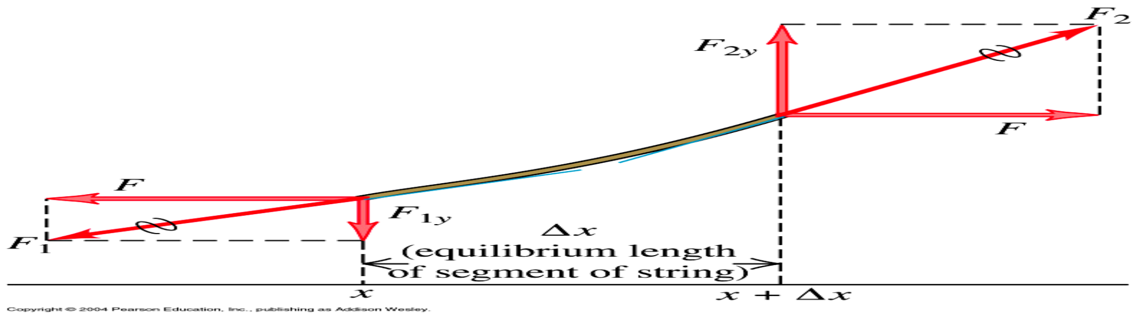
Giving the velocity as $v = \sqrt{\tau/\mu}$

Given this and f fixed by the driving source, the wavelength is $\lambda = v/f$

Wave Equation in One-Dimension:

The **wave equation** is a second order partial differential equation, which if it derives from the physics of a particular problem carries the implication that a propagating wave solution exists.

A wave equation for the transverse wave on a string results from Newton's 2nd law:



$$\sum F_y = F_2 \cdot Sin\,\theta_2 - F_1 \cdot Sin\,\theta_1 = ma = \Delta x \mu \frac{\partial^2 y}{\partial t^2}$$

In the limit as Δx becomes infinitesimal and for **small angles θ,**

$$F * [Tan(\theta_2) - Tan(\theta_1)] = \partial x \cdot \mu \frac{\partial^2 y}{\partial t^2}$$

$$F * [\partial Tan(\theta)] = \partial x \cdot \mu \frac{\partial^2 y}{\partial t^2}$$

$$F * \partial \cdot \frac{\partial y}{\partial x} = \partial x \cdot \mu \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

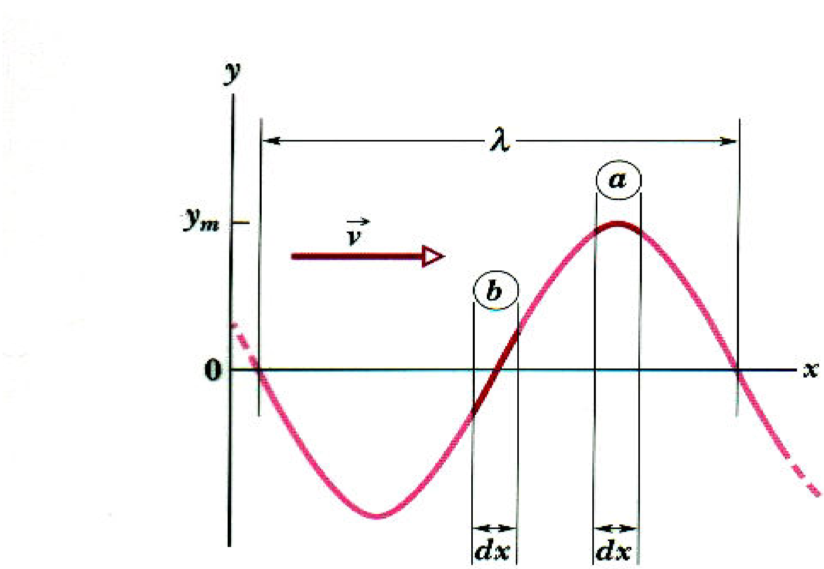
Energy Transport:

The energy of the wave is in the form of kinetic energy as string mass elements move with **transverse** velocity $u(x,t) = -\omega A Cos(kx - \omega t)$ and in the form of elastic potential energy as the string is stretched $PE_{Elastic} \propto [Stretching]^2$.

$$d(K) = \frac{1}{2}\mu dx[-\omega A Cos(kx - \omega t)]^2 = \frac{1}{2}A^2\omega^2 \mu dx * Cos^2(kx - \omega t)$$

Kinetic energy is a maximum if $(kx - \omega t)$ is zero. This is the point at which the string is crossing the equilibrium position and $y(x,t)$ is zero.

The string is also maximally extended when $y(x,t)$ is zero and therefore the potential energy is a maximum as the mass element **μdx** passes through equilibrium.



When the string mass element is at maximum amplitude its transverse velocity is zero and the string extension is zero. Kinetic and potential energies are zero at this location.

Time variation of the kinetic energy is:

$$\frac{dK}{dt} = \frac{1}{2}A^2\omega^2\mu\frac{dx}{dt} * Cos^2(kx - \omega t) = \frac{1}{2}A^2\omega^2\mu v Cos^2(kx - \omega t)$$

Averaging this to obtain the average rate of kinetic energy transported:

$$\left.\frac{dK}{dt}\right|_{AVE} = \frac{1}{2}A^2\omega^2\mu v\overline{Cos^2(kx - \omega t)}$$

The average of the cosine-squared function is:

$$\overline{Cos^2(kx - \omega t)} = \frac{1}{T}\int_0^T Cos^2(kx - \omega t)dt$$

Let $u = kx - \omega t$ $\rightarrow$ $du = -\omega dt$

$$\frac{1}{T}\int_0^T Cos^2(kx - \omega t)dt = \frac{-1}{\omega T}\int_{kx}^{kx-\omega T} Cos^2(u)du = \frac{-1}{\omega T}\int_{kx}^{kx-\omega T}\frac{1 + Cos2u}{2}du$$

$$\overline{Cos^2(kx - \omega t)} = \frac{-1}{\omega T}\left(\frac{u}{2} - \frac{Sin(2u)}{4}\right)_{kx}^{kx-\omega T} = \frac{1}{2}$$

$$\left.\frac{dK}{dt}\right|_{AVE} = \frac{1}{4}A^2\omega^2\mu v$$

The average rate for potential energy transport equals the rate of kinetic energy transport.

$$\left.\frac{dK}{dt}\right|_{AVE} = \left.\frac{d(PE)}{dt}\right|_{AVE}$$

The **average power transmitted** by the wave is then:

$$P_{AVE} = \frac{1}{2}A^2\omega^2\mu v$$

The dependence here is:

μ, v Depend on the material and the string tension.

y_m, ω Depend on the driving force that produces the wave.

The energy transported is proportional to the square of the wave amplitude.

Intensity

Intensity is the energy crossing perpendicularly through a unit surface area per unit time.

$$I = \frac{P}{A_\perp}$$

In **one dimension**, such as with a transverse wave moving along a rope, the unit area is fixed in time [cross-section of the rope] and both the intensity of the wave and its amplitude are constant neglecting losses.

For a point source in a **3-D isotropic medium**, waves propagate outward spherically and:

$$I = \frac{P}{4\pi r^2}$$

r is distance from the source. This falls off as $\mathbf{1/r^2}$

With power = energy / time, intensity is proportional to the square of amplitude $\mathbf{A^2}$:

$$I \propto A^2$$

Further **intensity is proportional to $\mathbf{1/r^2}$**, such that

$$\frac{1}{r^2} \propto A^2$$

Amplitude falls off as $\mathbf{1/r}$ away from the source. [No losses]

Lastly, since the power of the wave is: $$P_{AVE} = \frac{1}{2} A^2 \omega^2 \mu v$$

Intensity is proportional to $\mathbf{f^2}$ for mechanical waves. $$I \propto f^2$$

Superposition / Interference / Phasors

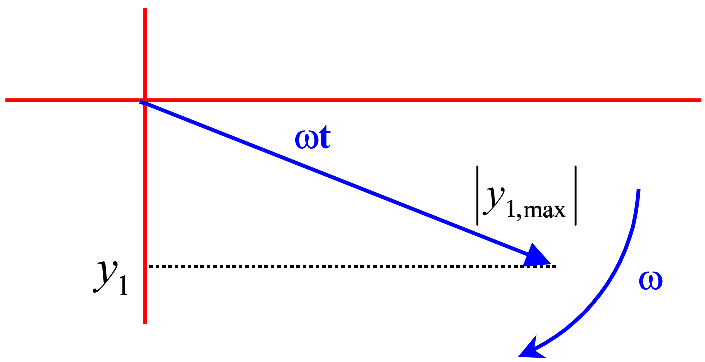
The principle of **superposition** states that for two or more waves overlapping in the same region of space, the resulting wave in that region can be found by an algebraic sum of the individual waves.

Depending on **phase differences** between the waves (i.e., the points in time at which each wave crests may vary) the resulting wave may have an amplitude greater than any of the incident waves or it may be less than the incident wave amplitudes and possibly zero.

Phasor Addition:

Phasors are vectors with magnitude equal to wave amplitude and which rotate clockwise about the origin with an angular velocity equal to the waves angular frequency **ω**.

Projection of a phasor onto the vertical axis gives the same analytic form as a propagating wave: $y_1(x,t) = y_{1,\max} Sin(kx - \omega t)$



The projection onto the vertical axis has range $-y_{1,\max} < y_1 < +y_{1,\max}$

To superpose a second wave of differing phase, its phasor is added to the first resulting in a wave having the form $y'(x,t) = y'_m Sin(kx - \omega t + \beta)$.

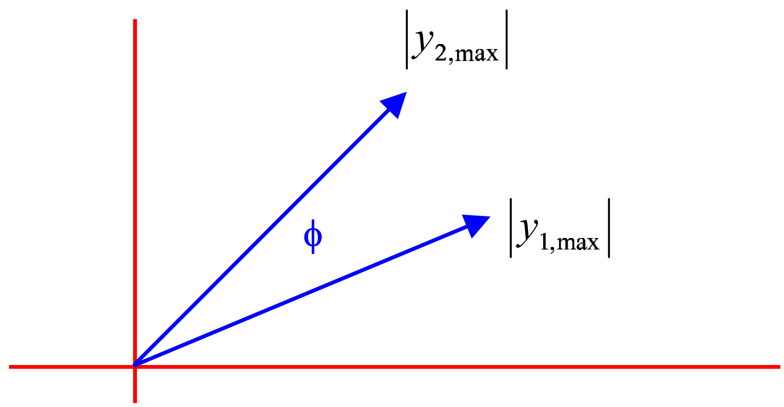
y'_m Is the amplitude of the superposition wave.

β Is the phase shift relative to $y_1(x,t) = y_{1,\max} Sin(kx - \omega t)$.

Example:

$$y_2(x,t) = y_{2,\max} Sin(kx - \omega t + \varphi)$$

ϕ > 0 ➔ the 2nd wave '**lags**' the 1st wave. **ϕ < 0 ➔** the 2nd wave '**leads**' the 1st wave

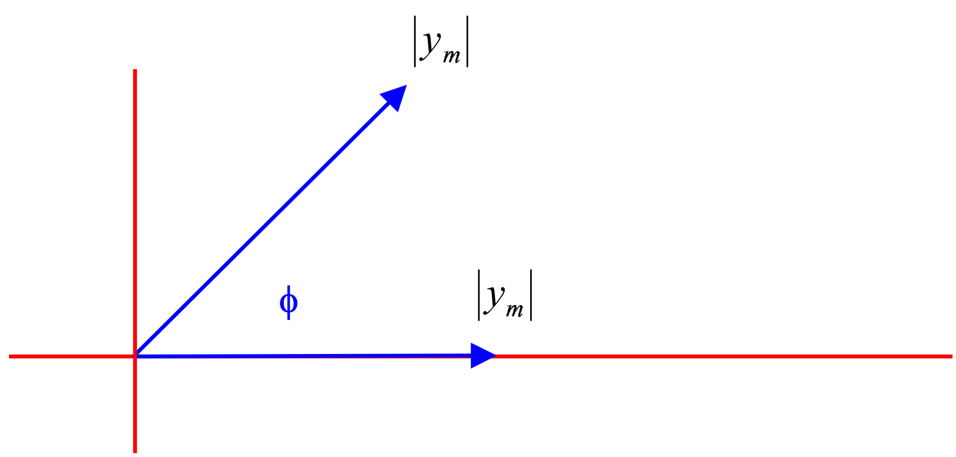


Consider interference between two waves with the **same amplitude and wavelength moving in the same direction**:

$$y_1(x,t) = y_m Sin(kx - \omega t)$$

$$y_2(x,t) = y_m Sin(kx - \omega t + \varphi)$$

Orienting the first Phasor along the horizontal simplifies the vector addition problem



The resulting wave is: $y'(x,t) = y'_m \, Sin(kx - \omega t + \beta)$ Where

$$y'_m = \sqrt{\left[y_m(1 + Cos(\varphi)\right]^2 + \left[y_m Sin(\varphi)\right]^2}$$

$$y'_m = \sqrt{y_m{}^2(1 + 2Cos\varphi + Cos^2\varphi + Sin^2\varphi)} = y_m\sqrt{2(1 + Cos\varphi)}$$

Since $Cos^2\dfrac{\varphi}{2} = \dfrac{1 + Cos\varphi}{2}$

$$y'_m = 2y_m Cos\frac{\varphi}{2}$$

The phase of the superposition wave is

$$\beta = Tan^{-1}\left|\frac{y_m Sin\varphi}{y_m(1 + Cos\varphi)}\right| = Tan^{-1}\left|\frac{Sin\phi}{2Cos^2\frac{\varphi}{2}}\right| = Tan^{-1}\left|\frac{2Sin\frac{\varphi}{2}Cos\frac{\varphi}{2}}{2Cos^2\frac{\varphi}{2}}\right|$$

$$\beta = Tan^{-1}\left|Tan\frac{\varphi}{2}\right| = \frac{\varphi}{2}$$

The resulting superposition is:

$$y'(x,t) = y'_m\ Sin(kx - \omega t + \beta) = 2y_m Cos\frac{\varphi}{2}\left[Sin(kx - \omega t + \frac{\varphi}{2})\right]$$

The amplitude is a maximum if $\phi = 0$ corresponding to **fully constructive interference**.

The amplitude is zero for $\phi = \pi$ corresponding to **fully destructive interference**.

Intermediate destructive interference takes place when $0 < \phi < \pi$.

In terms of a shift in wavelength between the two waves, the fully constructive interference ➔ zero shift: $Shift = \frac{\varphi}{2\pi/\lambda} = 0$

For fully destructive interference the wavelength shift is: $Shift = \frac{\varphi}{2\pi/\lambda} = \frac{\lambda}{2}$.

As a numeric example, let $|y_m|$ = **9.8mm** and **ϕ = 100⁰**

Find amplitude, type of interference and wavelength offset between the two waves.

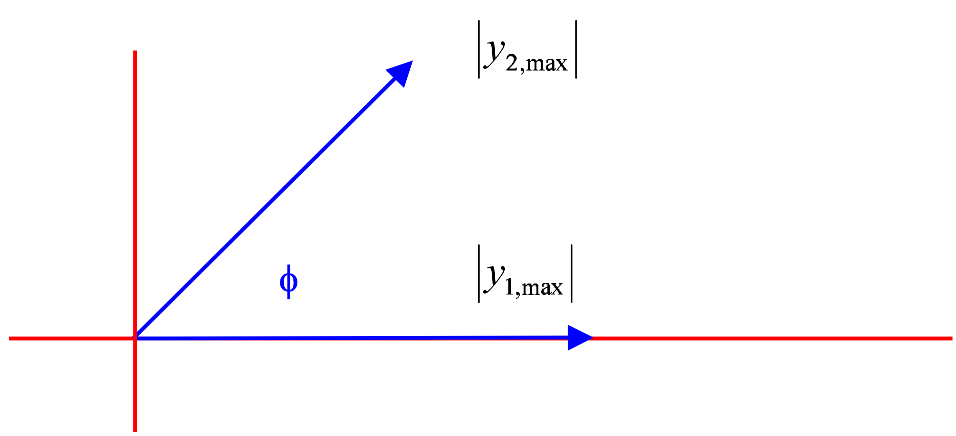
$$y'_m = 2y_m Cos\frac{\varphi}{2} = 2*(9.8mm)Cos\frac{100}{2} = 13mm \Rightarrow Intermediate$$

$$Shift = \frac{\varphi}{2\pi/\lambda} = \lambda * \frac{100*\frac{\pi}{180}}{2\pi} = \frac{1}{3.6}\lambda$$

Phasor example: Let

$$y_1(x,t) = (4.0mm)Sin(kx - \omega t)$$
$$y_2(x,t) = (3.0mm)Sin(kx - \omega t + \frac{\pi}{3})$$

Orienting wave 1 along the X-axis, and drawing the 2nd Phasor lagging at 60 degrees



$$|y'_{\max}| = \sqrt{(y_{1,\max} + y_{2,\max} Cos\frac{\pi}{3})^2 + (y_{2,\max} Sin\frac{\pi}{3})^2}$$

$$|y'_{\max}| = \sqrt{(4.0 + 3.0Cos\frac{\pi}{3})^2 + (3.0Sin\frac{\pi}{3})^2} = 6.1mm$$

$$\beta = Tan^{-1}\left|\frac{Vertical}{Horizontal}\right| = Tan^{-1}\left|\frac{3.0Sin\frac{\pi}{3}}{4.0 + 3.0Cos\frac{\pi}{3}}\right| = 0.44rad$$

$y'(x,t)$ **lags** $y_1(x,t)$ **by 0.44rad and the superposition is:**

$$y'(x,t) = (6.1mm)Sin(kx - \omega t + 0.44)$$

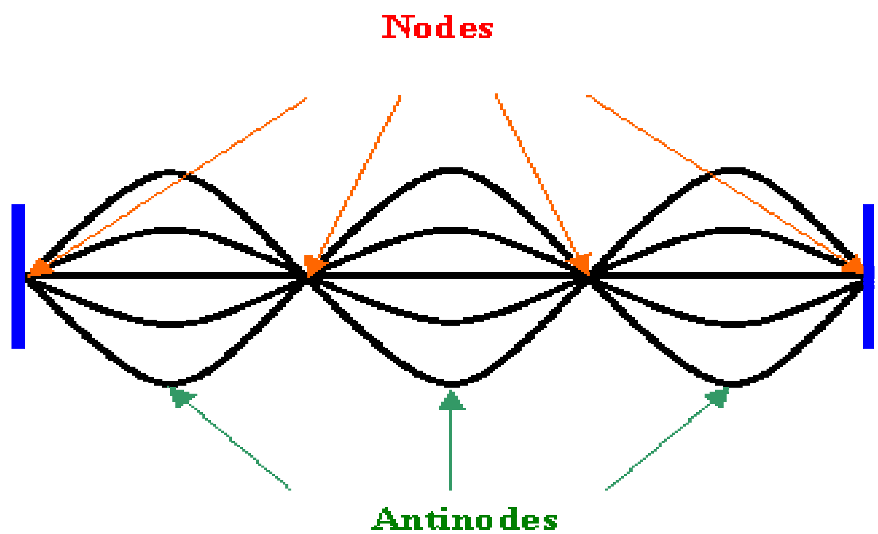
Standing Waves

Consider the interference between two traveling waves of equal wavelength and amplitude, but traveling in **opposite directions**.

In this case, a **standing wave** is created that oscillates in place without propagation hence the term 'standing'.

The interference results in points along the string where the string is stationary termed **nodes** and points which oscillate with maximum $\pm$ amplitude referred to as **antinodes**.

Note that nodes and antinodes are stationary. There is a local energy transfer between the nodes of a standing wave, but since no energy flows beyond each nodal point, standing waves do not transfer energy like traveling waves.



The location of node and antinode points may be determined as follows:

$$y_1(x,t) = y_m Sin(kx - \omega t)$$

$$y_2(x,t) = y_m Sin(kx + \omega t)$$

$$y'(x,t) = y_m (Sin(kx - \omega t) + Sin(kx + \omega t))$$

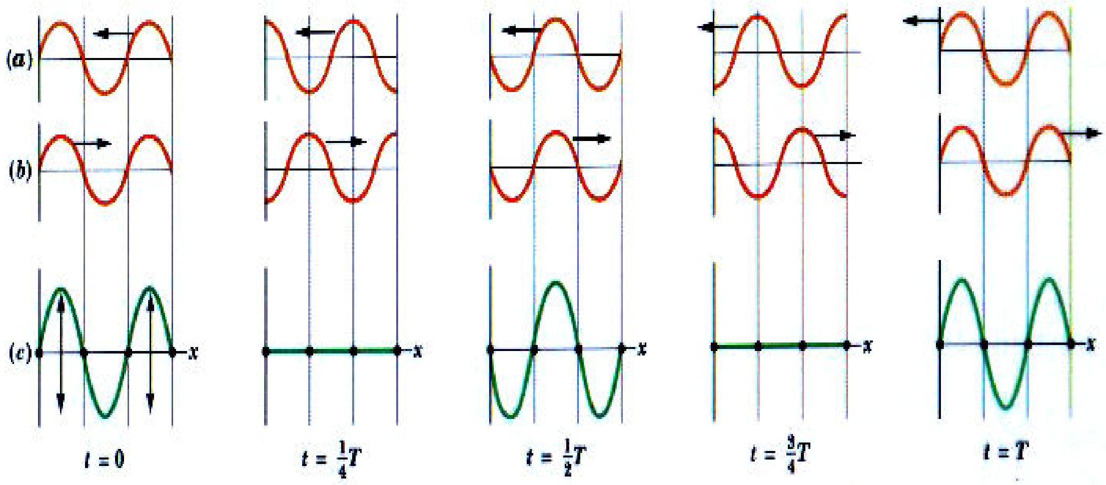
$$y'(x,t) = 2y_m Sin(kx) Cos(\omega t)$$

$$Amplitude = \left|2y_m Sin(kx)\right|$$

Nodes occur when $kx = n\pi$ **with n ={0,1,2…}**

Antinodes occur when $kx = n\dfrac{\pi}{2}$ **with n ={1,3,5…}**

Figure showing one cycle for a standing wave with 3-antinodes.



Transverse Standing Waves & Resonance

For a string between two fixed boundaries, **standing wave resonance** is setup along the string due to the boundary reflected waves interfering along the string.

The **frequency** at which a resonance occurs depends on the distance between the boundaries and the wave propagation velocity (string tension and mass per unit length).

A **half-integral number of wavelengths** contained within the boundary points will produce a resonant standing wave condition:

$$n\frac{\lambda}{2} = L$$ **With n ={1,2,3…}**

$$f = \frac{v}{\lambda} = n\frac{v}{2L}$$ **Where n ={1,2,3…}**

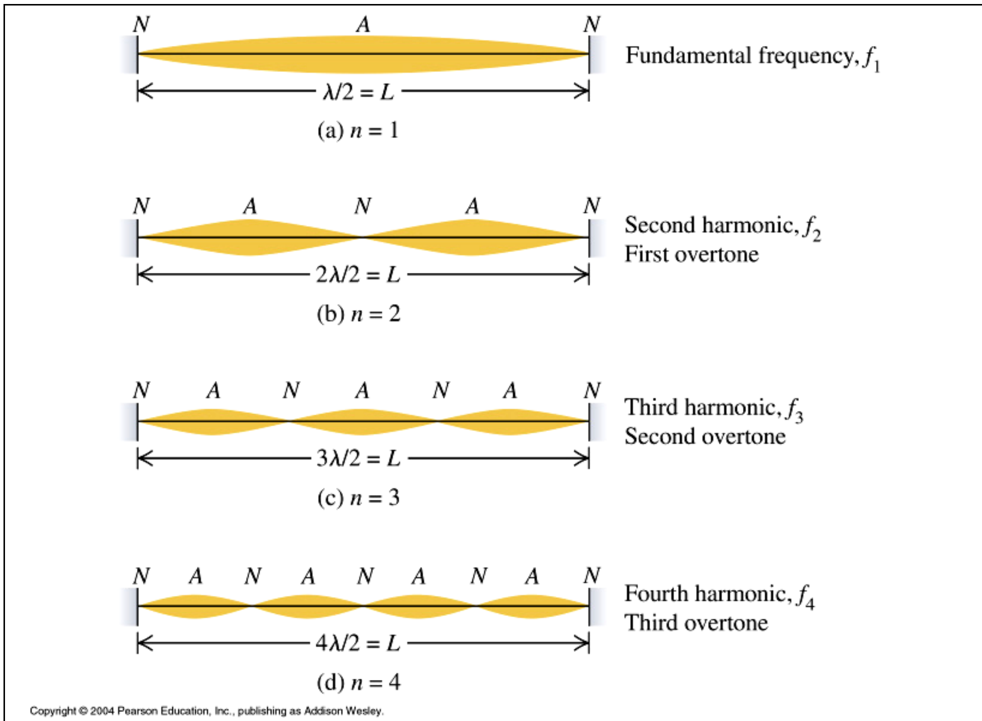
n = 1 ➔ Fundamental mode or 1st Harmonic

n = 2 ➔ 1st Overtone or 2nd Harmonic

n = 3 ➔ 2nd Overtone or 3rd Harmonic

The collection of all oscillating modes is referred to as a **harmonic series**.

Each particular resonance is called a **normal mode** of the system.



$$L = \frac{1}{2}\lambda_1$$

$$L = \frac{2}{2}\lambda_1$$

$$L = \frac{3}{2}\lambda_1$$

$$L = \frac{4}{2}\lambda_1$$

The fundamental frequency is $f_1 = \dfrac{v}{\lambda_1}$ where v is the wave velocity. For a string,

$$v = \sqrt{\tau/\mu}$$

The $\mathbf{n}^{\mathbf{th}}$ - harmonic has wavelength determined by the condition $L = n\dfrac{1}{2}\lambda_n$ and

the corresponding normal mode frequencies are $f_n = \dfrac{v}{\lambda_n} = nf_1$