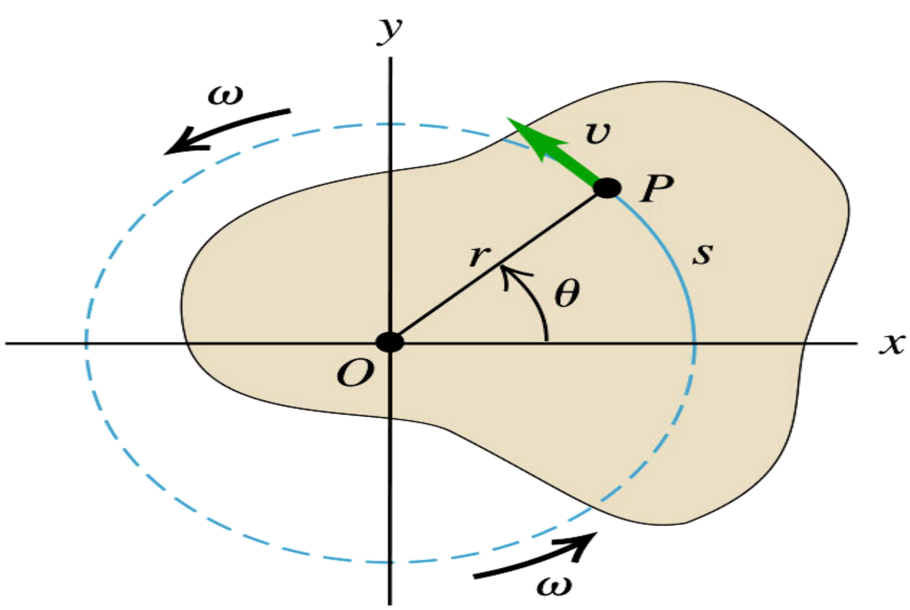
Rotational Motion

In Newtonian Mechanics Newton's Laws of Motion describe the dynamics of objects undergoing translation**.** This is also the case with **rotations**. To proceed, we appropriately redefine what is called a 'force', an 'acceleration', etc., in terms of angular quantities thereby arriving correctly at the analogous laws of motion for rotations.

Start with definitions and kinematics and then discuss the dynamics.

Consider only objects that are **rigid bodies**, and rotate about a fixed **axis of rotation** such that, for the moment, translations are ignored.

Consider a mass element in a **counter-clockwise** rotating object:



In sweeping through the **angular displacement** θ in an amount of time Δt, a point mass on the disk will rotate with an **angular velocity** $\omega = \dfrac{\Delta\theta}{\Delta t}$.

The **tangential velocity** for this mass is found using the following geometric relation for the arc length segment S :

$$S = r\Delta\theta$$ Where θ is in **radian measure** here for this equation to work.

$$V = \frac{\Delta S}{\Delta t} = r\frac{\Delta\theta}{\Delta t} = r\omega$$ Note the rigid body constraint.

The angular velocity ω has **units of radians per second** or s^{-1} since the radian is a dimensionless quantity.

Often you will also see ω in rpm

Conversion between degrees, revolutions, radians etc uses the factor:

$$1\,\mathrm{Rev} = 360^0 = 2\pi\_radians$$.

Recall that velocity of translation was a vector quantity and so was displacement. Here if we allow rotations in more than one dimension then our angular displacement cannot be considered a vector quantity for discrete rotations since vector associative properties are lost.

For one-dimensional rotations, **counter-clockwise rotations are taken as a positive angular displacement** and **clockwise rotations are negative angular displacements**.

Angular velocity is a vector with direction defined using the **right-hand-rule**:

Putting fingers out into the counterclockwise rotating direction of the rigid body above, then the thumb, pointing out of the page, gives the direction for the vector $\vec{\omega}$.

Angular Acceleration

If $\vec{\omega}$ is increasing in magnitude we have an angular acceleration:

$$\vec{\alpha} = \frac{\Delta\vec{\omega}}{\Delta t}$$

The direction of $\vec{\alpha}$ is, for a counterclockwise rotating object with an increasing angular velocity, given by the RHR as being out of the page.

If, for the counterclockwise rotation, $\vec{\omega}$ is decreasing in magnitude, then $\vec{\alpha}$ is back into the page.

The units of α are **radians per second squared** s^{-2}

$$S = r\theta$$

$$V_{Tangential} = r\omega$$

$$a_{Tangential} = r\alpha$$

Centripetal acceleration is $\dfrac{V^2}{r} = r\omega^2$

Angular velocity (speed):

$$\omega_{avg} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_f - \theta_i}{t_f - t_i}$$

Instantaneous angular velocity:

$$\omega = \frac{d\theta}{dt} \qquad \omega = \lim_{\Delta t -> 0} \frac{\Delta\theta}{\Delta t}$$

Angular acceleration:

$$\alpha_{avg} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{t_f - t_i}$$

Instantaneous angular acceleration:

$$\alpha = \frac{d\omega}{dt} \qquad \alpha = \lim_{\Delta t -> 0} \frac{\Delta\omega}{\Delta t}$$

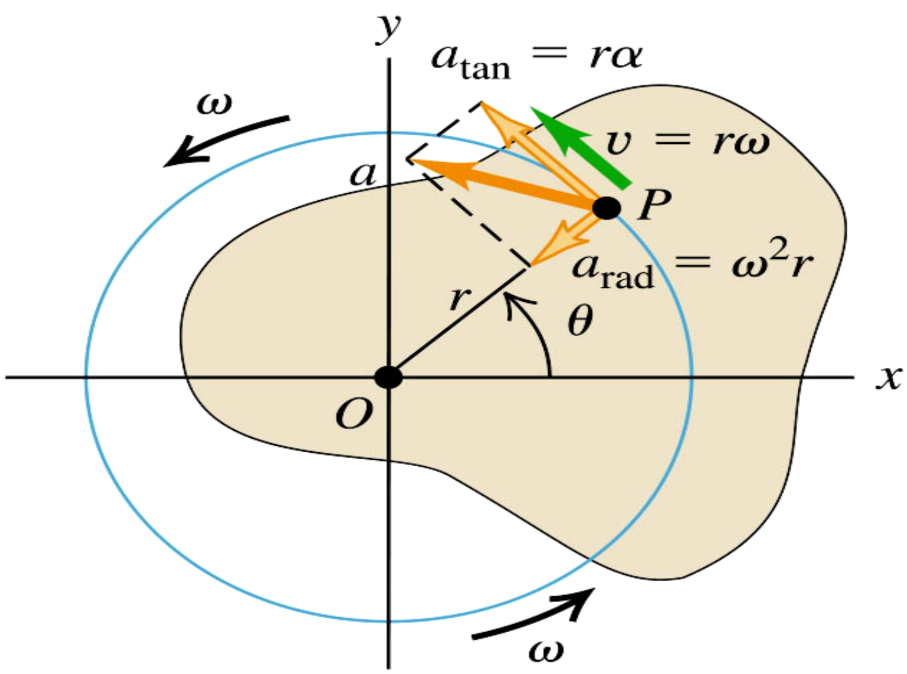
Translation variable ⇐⇒ rotation variable relations:

$$S = r\theta$$

$$V_{Tangential} = r\omega$$

$$a_{Tangential} = r\alpha$$

$$a_{Radial} = \frac{V^2}{r} = r\omega^2$$



Constant angular acceleration kinematic equations:

The kinematic equations are found either by substituting the above translation variable ←→rotation variable quantities into the translation kinematic equations, or by analogy:

$$v_f = v_0 + at \qquad \omega_f = \omega_0 + \alpha t$$

$$x_f = x_0 + v_0 t + \frac{1}{2}at^2 \qquad \theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$v_f^{\ 2} = v_0^{\ 2} + 2a\Delta x \qquad \omega_f^{\ 2} = \omega_0^{\ 2} + 2\alpha\Delta\theta$$

These equations can be used to acquire angular kinematic information about a problem. In conjunction with a rotational form of Newton's Second Law, the rotation kinematic parameters are used in much the same way as were the translation kinematic parameters.

Rotational Motion Dynamics

The rotational form of Newton's 2nd Law requires working definitions for 'angular forces' or **Torques** and for 'angular mass' i.e., **rotational inertia**.

Rotational inertia is an object's resistance to being angularly accelerated, and is unlike translation inertia in that when an object is rotating about an **axis of rotation**, it is usually an irregularly shaped or extended object that cannot be treated as point like.

Moment of Inertia; Discrete Objects

The moment of inertia for a **system of particles** with masses m_i each a perpendicular distance r_i away from the axis of rotation is defined as the sum of the product of these masses and the square of the perpendicular distance from the rotation axis:

$$I = \sum_i m_i r_i^2 \qquad I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + m_4 r_4^2$$

Moment of Inertia; Continuous Objects

In the discrete masses case we have the summation $I = \sum_i m_i r_i^2$. For a continuous extended mass, the summation becomes an integral $\sum_i m_i r_i^2 \to \int r^2 dm$ and integration is over infinitesimal mass elements in the rigid body.

Parallel Axis Theorem

The moment of inertia result for the slab about its corner axis could have been found also using the **parallel axis theorem**:

The moment of inertia of an object about an axis parallel to an axis of rotation through the object COM and located a perpendicular distance **d** from the COM axis is:

$$I_{Parallel\_Axis} = I_{COM} + Md^2$$

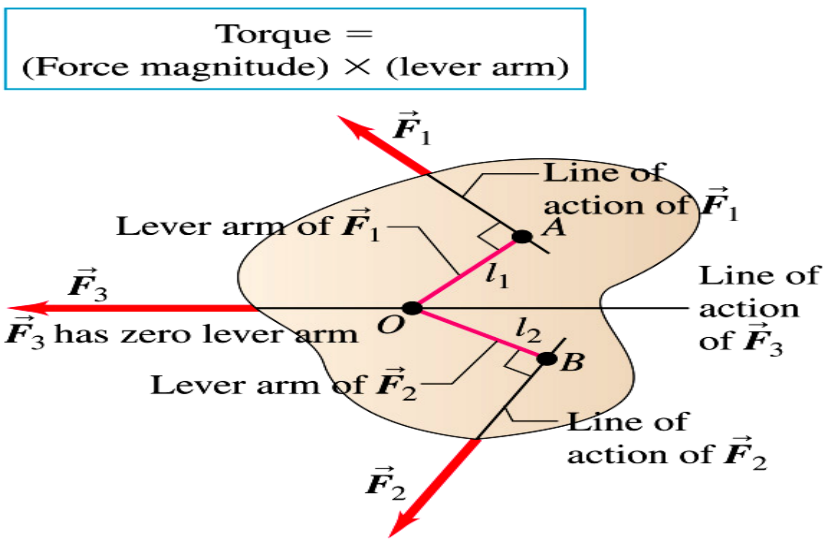
Rotational Force: Torque (Latin for 'Twist')

Consider the hinges of a door as a vertical axis of rotation for the door. In applying a force to the door (pushing on it for example), then if that force is directly on the hinges of the door there is no rotation. If, however, a **lever arm** is used and we push the door at the knob, then we can angularly accelerate the door about its hinge axis of rotation.

Torque is the product of external force with lever arm distance from the axis of rotation:

$$\tau = F_{ext}\{L_{\perp}\}$$

The lever arm distance is the perpendicular distance for the axis of rotation to the line of sight of the applied force.

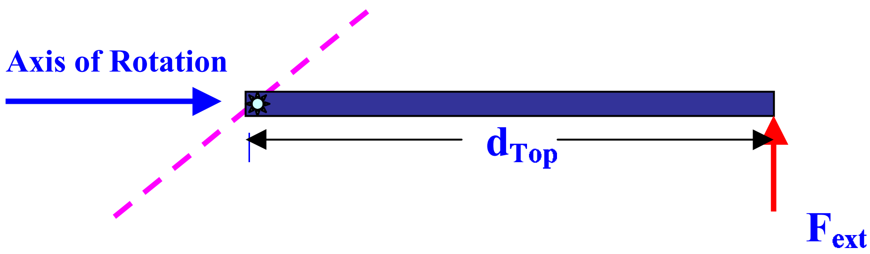


Before looking at some examples, notice:

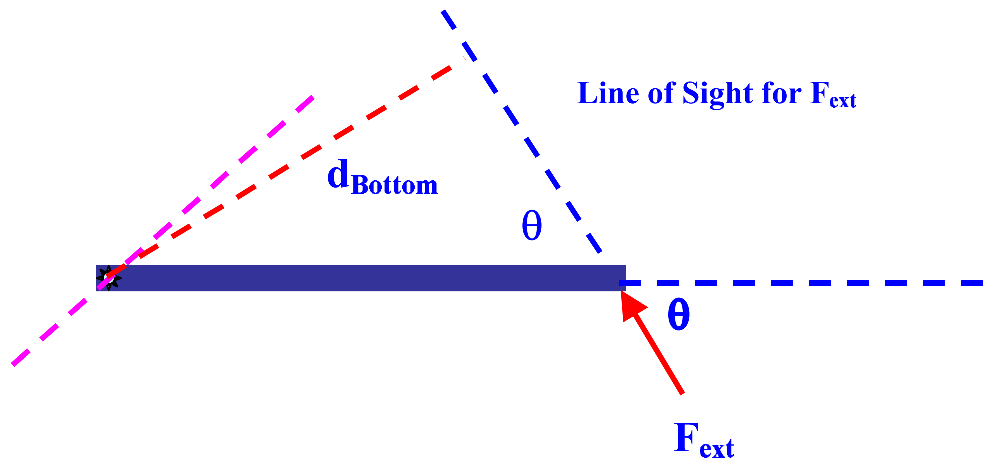
1) **SI units of Torque are N-m, which is not called a Joule in this context.**

2) **Torques are vector quantities whose magnitude and sign are given by the cross product definition of torque** $\vec{\tau} = \vec{r} \times \vec{F}$ **and RHR.**

3) **From an X-Y plane perspective and assuming the object is at rest, then forces that would result in counterclockwise rotations of an object correspond to positive torques and forces resulting in clockwise rotations are taken as negative torques. This represent a +/- sign convention for torques.**



$$\tau = +F_{ext}\left\{d_{Top}\right\}$$



$\mathbf{d_{Bottom}}$ is the perpendicular distance from the axis of rotation to the line of sight of F_{ext} .

$$\tau = +F_{ext}\left\{d_{Bottom}\right\}$$

Notice the following:

1) **d_{Top} is not equal to d_{Bottom}**

1) **$\tau_{Top} > \tau_{Bottom}$**

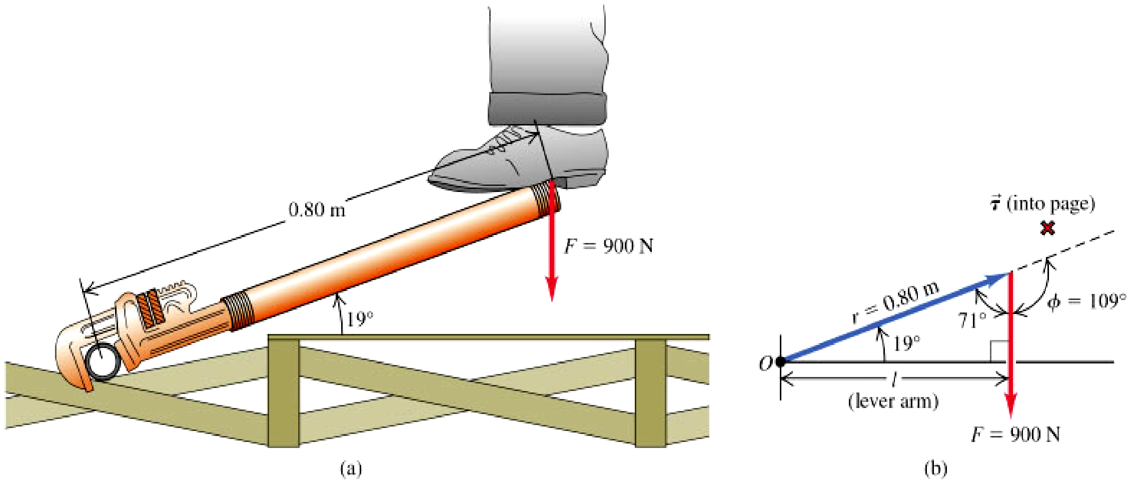
3) **$F_{Ext\_Top} = F_{Ext\_Bottom}$ in magnitude.**

4) **$d_{Top}\,(\mathrm{Sin}(\theta)) = d_{Bottom}$**

Changing the orientation of F_{ext} relative to the beam axis changes the lever arm and reduces the torque by a factor **Sin(θ)**. If **θ** is zero degrees then τ is zero.

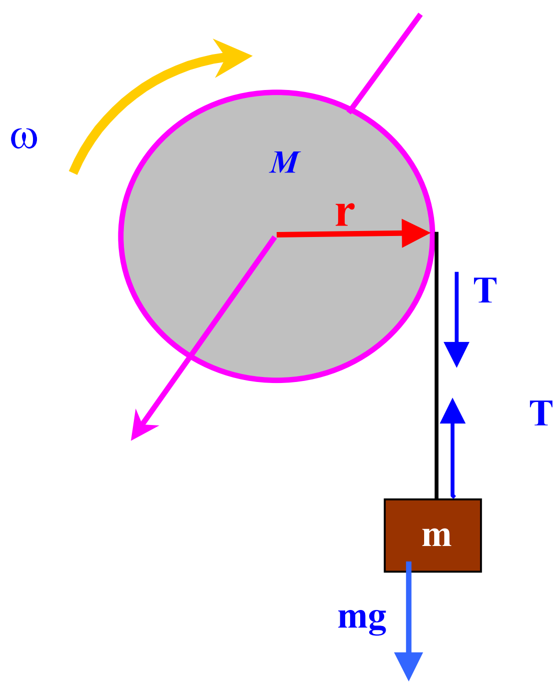
Torque and Vector Cross-Products

$$\vec{\tau} = \vec{r} \times \vec{F} = |\vec{r}||\vec{F}| \, Sin(\theta)$$ **RHR**



The rotational form of **Newton's 2nd Law** is: $\sum \tau_{Net\_Ext} = I\alpha$

Example: Let mass **m** hang from a rope of negligible mass wound about a pulley of mass **M**. Find the acceleration of **m**:



$$T - mg = -ma$$

$$T = mg - ma$$

For the pulley mass M,

$$\tau = rT\sin 90^0 = I\alpha$$

$$T = I\frac{\alpha}{r} = I\frac{a}{r^2}$$

Equating the two and using $I_{Disk} = \frac{1}{2}Mr^2$ we can write

$$I\frac{a}{r^2} + ma = mg$$

$$a(m + \frac{1}{2}M) = mg$$

$$a = \frac{mg}{m + \frac{1}{2}M}$$ **For the zero mass pulley we get $a = g$ as expected.**

$$T = I\frac{a}{r^2} = \frac{1}{2}M\left\{\frac{mg}{m + \frac{1}{2}M}\right\}$$

$$\alpha = \frac{a}{r} = \frac{1}{r}\frac{mg}{m + \frac{1}{2}M}$$

After falling a time t the angular velocity is $\omega = \alpha t$ and the disk kinetic energy is:

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}I\{\alpha t\}^2 = \frac{1}{4}M\left\{\frac{mg}{m + \frac{1}{2}M}\right\}^2 t^2$$

Equilibrium

For the static or equilibrium applications, require that all translation accelerations and rotational accelerations be identically zero:

$$\sum \tau_{Net\_Ext} = 0$$

$$\sum F_{Net\_Ext,X} = 0$$

$$\sum F_{Net\_Ext,Y} = 0$$

$$\sum F_{Net\_Ext,Z} = 0$$

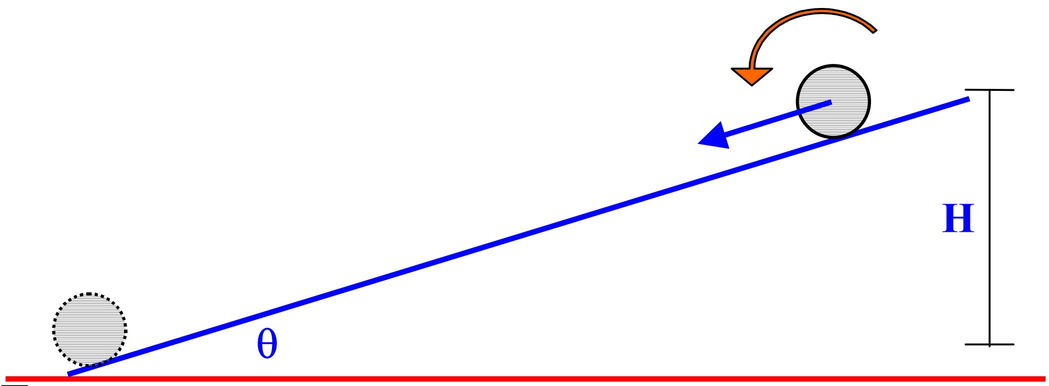
These equations of **static equilibrium** supply us with several relations from which we may determine the conditions under which the system remains static.

Notice that a system may be in motion with a uniform velocity and still satisfy these conditions. This is **non-static equilibrium**.

Rolling Motion

A disk **rolls without slipping** down an inclined plane. Determine its final velocity at the bottom of the plane first using energy methods and then with Newton's 2nd Law.

The constraint of rolling without slipping allows us to write for the disk displacement along the incline $X = S = r\Delta\theta$. Velocity of translation is $V_{Com} = \omega \bullet r$



$$E_0 = E_f$$

$$mgH = \frac{1}{2}mV^2 + \frac{1}{2}I\omega^2 \qquad V_{Com} = \omega \bullet r$$

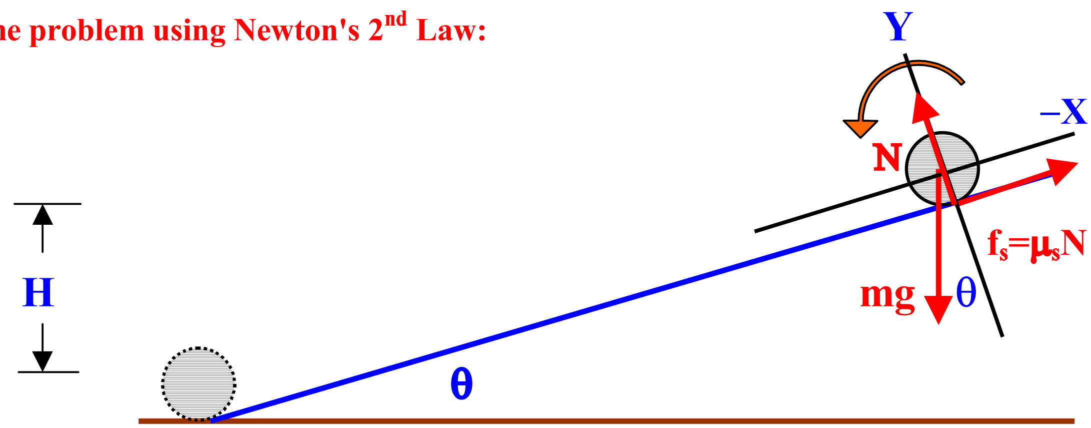
$$mgH = \frac{1}{2}mV^2 + \frac{1}{2}I\left\{\frac{V}{r}\right\}^2$$ Which upon using $I_{Disk} = \frac{1}{2}Mr^2$

$$mgH = \frac{1}{2}mV^2 + \frac{1}{2}\frac{1}{2}mr^2\left\{\frac{V}{r}\right\}^2$$

$$mgH = V^2\left\{\frac{1}{2}m + \frac{1}{4}m\right\}$$

$$V = \sqrt{\frac{4}{3}gH}$$

Same problem using Newton's 2nd Law:



The torque at the point of contact is:

$$\tau = rf_s = I\alpha \qquad\qquad rf_s = I\frac{a}{r}$$

In X: $$mgSin(\theta) - f_s = ma$$

$$mgSin(\theta) - I\frac{a}{r^2} = ma$$

Using $I_{Disk} = \frac{1}{2}Mr^2$

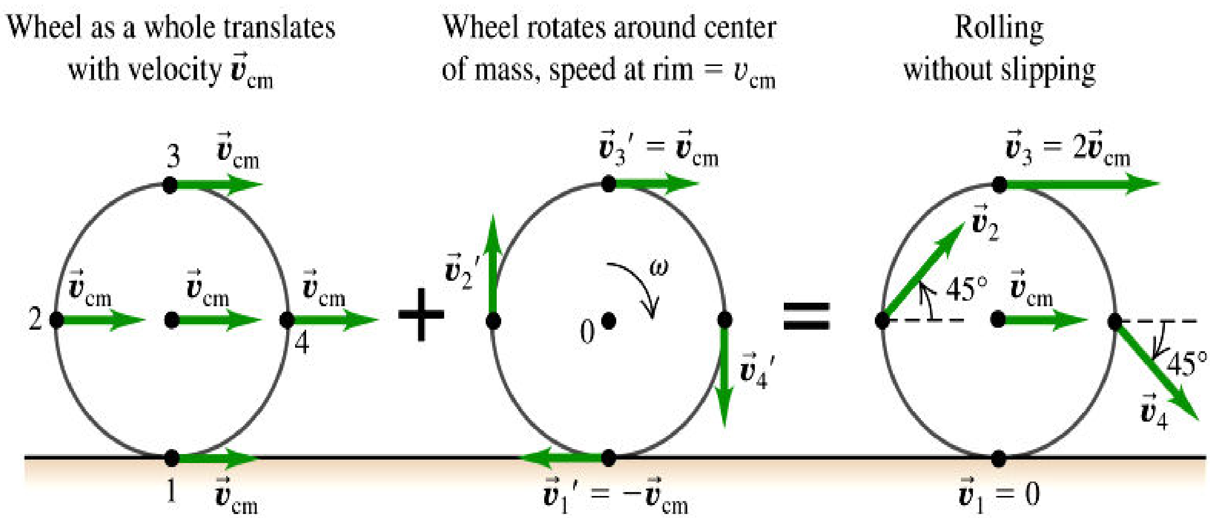
$$mgSin(\theta) = ma + \frac{1}{2}ma \qquad\qquad a = \frac{2}{3}gSin(\theta)$$

$$V_f^{\ 2} = V_0^{\ 2} + 2a\Delta X = 0 + \frac{4}{3}gSin(\theta)\frac{H}{Sin(\theta)}$$

$$V_f = \sqrt{\frac{4}{3}gH}$$

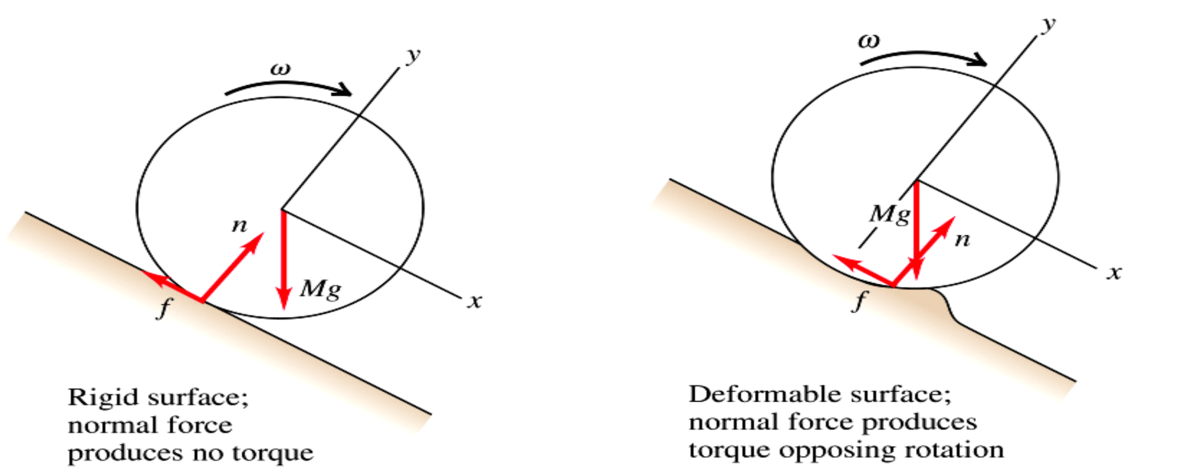
Combined Translation and Rotation:

For rolling without slipping rim velocities due to the rotation adds to the wheel translation COM velocity:



Rolling Friction

If surfaces in contact for rolling motion are not perfectly smooth, then the normal force has a variable location that may be off-axis and therefore produce a torque on the wheel.

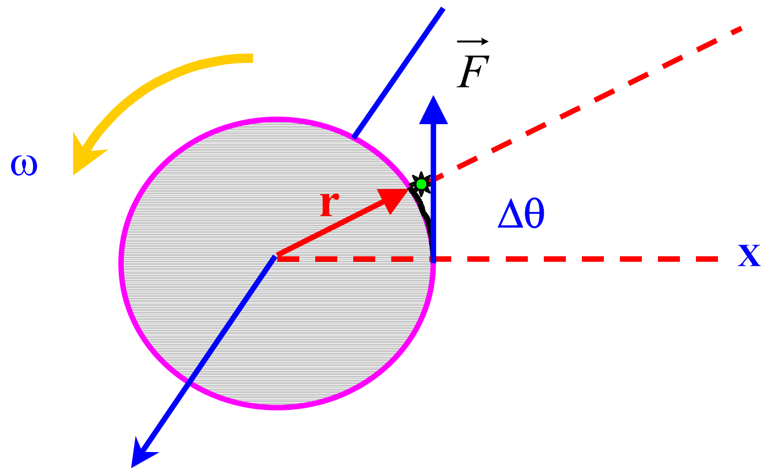


(a) (b)

Coefficients of rolling friction are approximately 10^{-2} for rubber tires on pavement.

Work-Kinetic Energy Theorem for Rotations:

The work-kinetic energy theorem also includes rotational kinetic energies and torques:



$$W_{F,ext} = \Delta K = \frac{1}{2} I \omega_f^{\ 2} - \frac{1}{2} I \omega_0^{\ 2}$$

Force $\vec{F}$ is applied over an arc length $S = r\Delta\theta$ so the left hand side of the theorem is $\vec{F} \cdot \vec{r}\Delta\theta$. Further $\vec{F}$ and $\vec{r}$ are at right angles implying the torque

$\vec{\tau} = \vec{r} \times \vec{F}$ is just equal to $|\vec{r}||\vec{F}|$ and the work done in the rotational case is:

$$W = \tau\Delta\theta = \tau(\theta_f - \theta_i)$$ For a constant torque.

The power or rate at which $\vec{F}$ is doing work is:

$$P = \frac{\Delta W}{\Delta t} = \tau \frac{\Delta\theta}{\Delta t} = \tau\omega$$

$$P = \vec{\tau} \bullet \vec{\omega}$$ Recall the $\vec{F} \bullet \vec{V}$ form of translation dynamics.

Angular Momentum

The **angular momentum** of an object about a rotation axis is $\vec{L} = I\vec{\omega}$

Newton's 2nd Law is; $\vec{F} = \frac{\Delta\vec{P}}{\Delta t}$

The rotation equation is $\vec{\tau} = \frac{\Delta\vec{L}}{\Delta t}$.

If all external torques are identically zero, then $\vec{L} = I\vec{\omega}$ is a constant in time.

Meaning **angular momentum is conserved:** $\vec{L_0} = \vec{L_f}$

In the absence of net external torques, angular momentum is conserved.

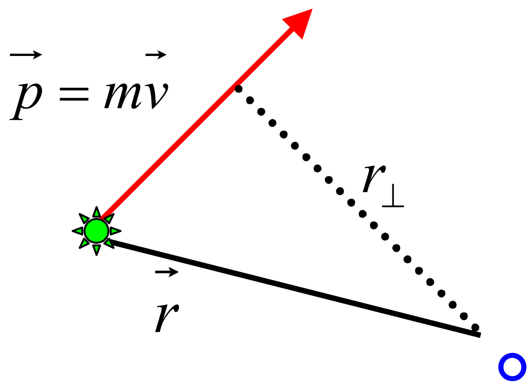
$$I_0\vec{\omega_0} = I_f\vec{\omega_f}$$

Since $\vec{\tau} = \vec{r} \times \vec{F}$ then $\vec{\tau}\Delta t = \vec{r} \times \vec{F}\Delta t$

And $\Delta\vec{L} = \vec{r} \times \Delta\vec{P}$

Giving $\vec{L} = \vec{r} \times \vec{P}$ as another way to calculate angular momentum.

Angular momentum is the product of the particles momentum $\vec{p} = m\vec{v}$ multiplied by the perpendicular distance from the point of observation to the momentum vector line-of-sight.



For systems of particles, we have the vector sum $\vec{L} = \vec{l}_1 + \vec{l}_2 + \ldots + \vec{l}_n$

$$\frac{\Delta \vec{L}}{\Delta t} = \frac{\Delta}{\Delta t}\left\{\vec{l}_1 + \vec{l}_2 + \ldots + \vec{l}_n\right\} = \sum_{i=1}^{n} \vec{\tau}_{net,i}$$

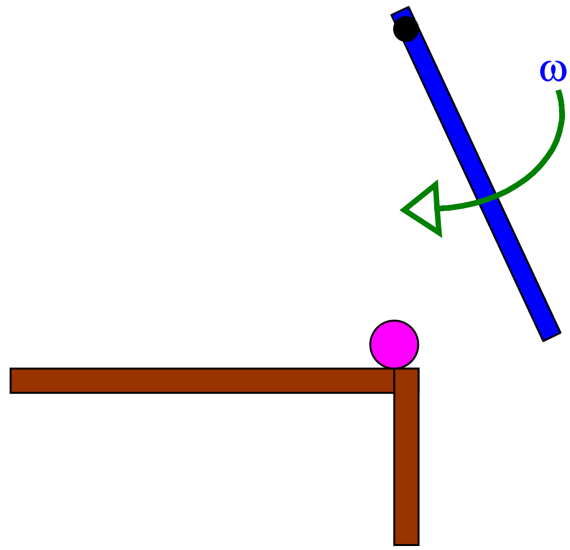
The rate of change of system angular momentum is the vector sum of torques on its individual particles.

$$\vec{\tau}_{net} = \frac{\Delta \vec{L}}{\Delta t}$$

Example:

A rod of mass **1 Kg** and length **0.6m** has a rotational inertia of **0.12 kgm^2**. Pivoting about its top end, the bottom end of the rod collides in a completely inelastic collision with a **0.2 Kg** putty wad. If the angular velocity before the collision was **2.4 rad/s**, then find the final angular speed of the system after the collision.

Since the pivot point is fixed, external forces from the collision act there implying a zero lever arm for torques about that point. Therefore, we conserve angular momentum:



$$L_0 = L_f$$

$$I_0\omega_0 = I_f\omega_f$$

$$(0.12 kg \bullet m^2) \bullet 2.4 s^{-1} = [(0.12 + 0.20(0.6)^2) kg \bullet m^2]\omega_f$$

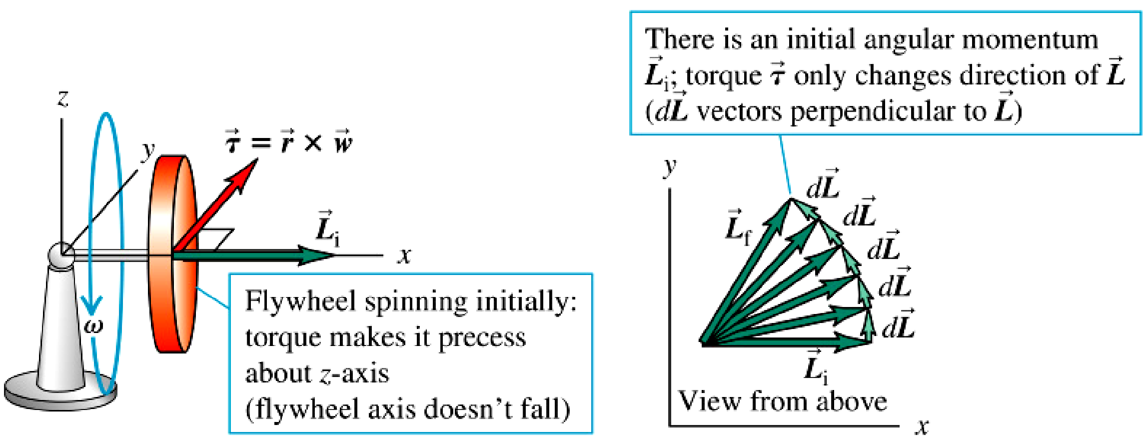
$$\omega_f = \frac{0.288}{0.192} s^{-1}$$

$$\omega_f = 1.5 s^{-1}$$

Gyroscopes and Precession

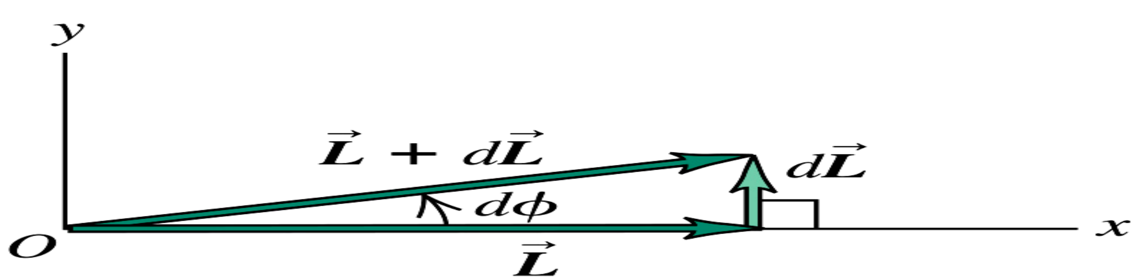
With an initial angular momentum vector along the **x-axis**, the $\overrightarrow{dL}$ produced by torque from the weight vector sets up a precession about the **z-axis**.

Notice that if the gyroscope is not initially spinning the weight vector torque rotates the gyroscope about the **y-axis** into the tabletop.



(a) (b)

The precession rate Ω is:



$$\Omega = \frac{d\phi}{dt} = \frac{\left|\overrightarrow{dL}\right| / \left|\vec{L}\right|}{dt} = \frac{\tau}{L} = \frac{wr}{I\omega}$$

Rapidly spinning gyroscopes therefore have slow precession rates.