Linear Momentum

The **linear momentum** of an object is the product of its mass with its velocity vector:

$$\vec{P} = m\vec{V}$$ **SI Units here are** $kg * m / s$

With a time rate of change, Newton's 2nd Law using the linear momentum vector is:

$$\frac{\Delta \vec{P}}{\Delta t} = M\frac{\Delta \vec{V}}{\Delta t} = M\vec{a} = \vec{F}_{Net\_Ext}$$

If $\vec{F}_{Net\_Ext} = 0$ then:

$$\frac{\Delta \vec{P}}{\Delta t} = 0$$ **The momentum vector is a constant in time**.

In the absence of net external forces, total linear momentum is conserved:

$$\vec{P}_i = \vec{P}_f$$ **This is the law of conservation of linear momentum.**

Further, since $\frac{\Delta \vec{P}}{\Delta t} = \vec{F}_{Net\_Ext}$ is a vector equation it is possible that momentum conservation holds in one coordinate direction but not all coordinate directions.

For all coordinates where the external net force identically zero we can write:

$$P_{iX} = P_{fX} \qquad P_{iY} = P_{fY} \qquad P_{iZ} = P_{fZ}$$

In a closed system of n-particles:

$$\overrightarrow{P}_{Total} = \overrightarrow{P_1} + \overrightarrow{P_2} + \ldots + \overrightarrow{P_n}$$

$$M\overrightarrow{V}_{COM} = m_1\overrightarrow{v_1} + m_2\overrightarrow{v_2} + \ldots + m_n\overrightarrow{v_n}$$

$$\overrightarrow{P}_{Total} = M\overrightarrow{V}_{COM} \qquad M = \sum_i m_i$$

The total system linear momentum is the system mass times $\overrightarrow{V}_{COM}$

Systems of Particles

When external forces are applied to a **system of particles**, for example a continuous extended object or a many particle system, then the system dynamics are described in terms of motion **of the center of mass** and motion **about the center of mass**.

The **center of mass** for a system of particles is the point that moves as though all of the system mass were concentrated there and all external forces were applied there.

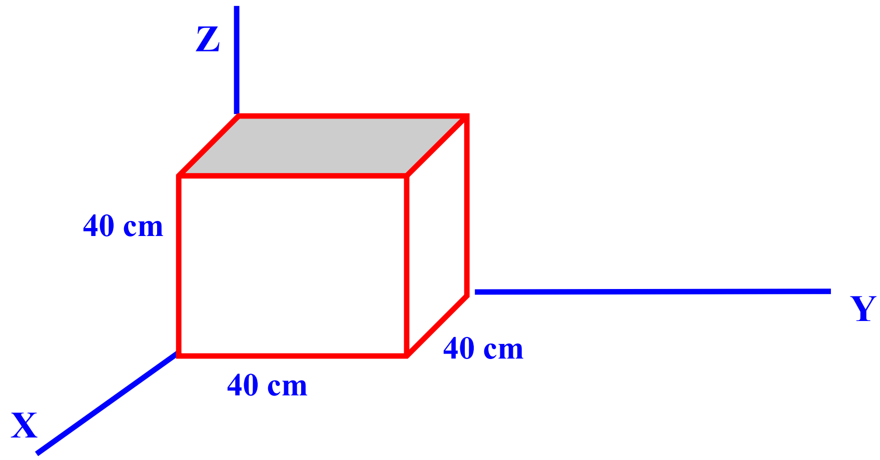
E.g., Traveling circus clowns juggling bowling pins flip the pins head top over bottom back and forth. The **COM** point in each pin follows a parabolic trajectory and motion of all other points are referenced using their rotation about the center of mass.

The center of mass definitions for a **discrete system of many particles** are:

$$X_{COM} = \frac{\sum_i m_i x_i}{\sum_i m_i} \qquad Y_{COM} = \frac{\sum_i m_i y_i}{\sum_i m_i} \qquad Z_{COM} = \frac{\sum_i m_i z_i}{\sum_i m_i}$$

Note the COM point **does not** necessarily lie within the object:

E.g., a 5-sided cube of steel square plates each of mass M and 1600 sq. cm in area:



The cube is symmetric in **X** and **Y** but not **Z** (**no top plate**). Expect **X** and **Y** COM to be central and the **Z** COM below 20 cm since the cube is biased toward the bottom plate. Take the cube to be a system of **discrete masses** located at the plate center points:

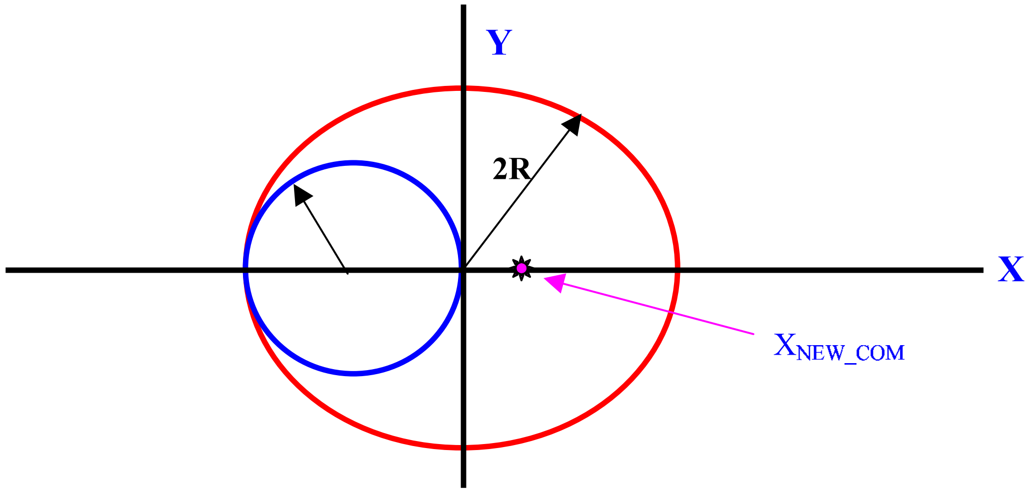
Mass M **@ (20,0,20)**
Mass M **@ (20,20,0)**
Mass M **@ (40,20,20)**
Mass M **@ (0,20,20)**
Mass M **@ (20,40,20)**

$$X_{COM} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{M(100)}{5M} = 20cm$$

$$Y_{COM} = \frac{\sum_i m_i y_i}{\sum_i m_i} = \frac{M(100)}{5M} = 20cm$$

$$Z_{COM} = \frac{\sum_i m_i z_i}{\sum_i m_i} = \frac{M(80)}{5M} = 16cm$$

Consider next a circular plate with a slug punched out from left of center:



With the slug removed, X_{COM} shifts in the +X direction to the purple star X_{NEW_COM}

By Symmetry $Y_{COM} = 0$ **With the slug in place $X_{COM} = 0$:**

$$X_{COM} = \frac{M_{Slug} X_{Slug} + M_{Plate-slug} X_{NEW\_COM}}{M_{Slug} + M_{Plate-Slug}} = 0$$

$$X_{NEW\_COM} = \frac{-M_{Slug} X_{Slug}}{M_{Plate-Slug}} = \frac{-\rho_{Slug} V_{Slug} X_{Slug}}{\rho_{Plate-Slug} V_{Plate-slug}}$$

$$X_{NEW\_COM} = \frac{-V_{Slug}}{V_{Plate-slug}}(-R)$$

$$X_{NEW\_COM} = \frac{A_{Slug}}{A_{Plate-slug}} R = \frac{\pi R^2}{\pi (2R)^2 - \pi R^2} R$$

$$X_{NEW\_COM} = R/3$$

Newton's Second Law for Systems of Particles

The COM point is a point that moves like a particle with mass equal to the total system mass. When there is a net external force Newton's 2nd Law applies:

For a system of n-particles,

$$\vec{r}_{COM} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} \qquad M\vec{r}_{COM} = m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n$$

$$M\vec{V}_{COM} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots + m_n\vec{v}_n$$

$$M\vec{a}_{COM} = m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots + m_n\vec{a}_n$$

$$M\vec{a}_{COM} = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_n = \sum External\_Forces + \ldots$$

$$\sum \vec{F}_{Net\_Ext} = M\vec{a}_{COM}$$

$$\sum F_{Net,X} = Ma_{COM,X}$$

$$\sum F_{Net,Y} = Ma_{COM,Y}$$

$$\sum F_{Net,Z} = Ma_{COM,Z}$$

1) $\vec{F}_{Net\_Ext}$ Refers to **external forces only** on the closed system.

2) M Is the total system mass that is assumed constant, i.e., **system is closed.**

3) $\vec{a}_{COM}$ Is acceleration of the center of mass point and **does not necessarily** refer to acceleration for one of the particles in the system.

Consider a billiard ball collision where ball 1 with mass M_1 is at rest and ball 2 with mass M_2 moves toward ball 1 colliding **elastically**. We expect the two balls of equal mass to move off in the forward direction. Since $\vec{F}_{Net\_Ext} = 0$ during the collision we have $\vec{a}_{COM} = 0$ implying that $\vec{V}_{COM} = const.$ throughout the process and

$$\vec{r}_{COM} = linear\_in\_time.$$

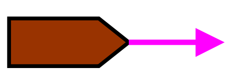
Systems of Varying Mass

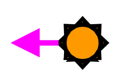
When the system is losing or acquiring mass, we extend our definition of the isolated system to include masses **coming into** or **leaving** the original system.

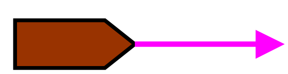
A good example is Rocket Propulsion. Exhausted fuel products constitute mass removed from the primary system namely the rocket.

Extending our system to include the exhaust products and using $\vec{F}_{Net\_Ext} = 0$, the conservation of linear momentum leads to:

Before engine burn: **After burning & exhausting spent fuel**







Mass M, Velocity V **Mass -dM, Velocity U** **Mass M+dM Velocity V+dV**

Note here that dM is negative and -dM a positive quantity.

Conserving Linear Momentum in One-Dimension,

$$MV = -dMU + (M + dM) * (V + dV)$$

$$MV = -dMU + MV + MdV + VdM + dM * dV$$

$$MdV + VdM = UdM$$

The relative velocity between the ship and the exhaust products is

$$V_{relative} = V + dV - U$$

$$MdV + VdM = (V + dV - V_{relative}) * dM$$

$$MdV = -V_{relative} * dM$$

$$-\frac{dM}{dt} V_{relative} = M \frac{dV}{dt}$$

The mass fuel consumption rate is $R = -\dfrac{dM}{dt}$

The 1st Rocket Equation. $RV_{relative} = Ma$

$$RV_{relative} = Thrust = T$$

$$T = Ma$$

Starting from

$$\frac{-dM}{dt} V_{rel} = M \frac{dV}{dt}$$

$$\frac{-dM}{M} V_{rel} = dV$$

$$V_{rel} \int \frac{-dM}{M} = \int dV$$

$$V_f - V_i = -V_{rel} (\ln M) |_{M_i}^{M_f}$$

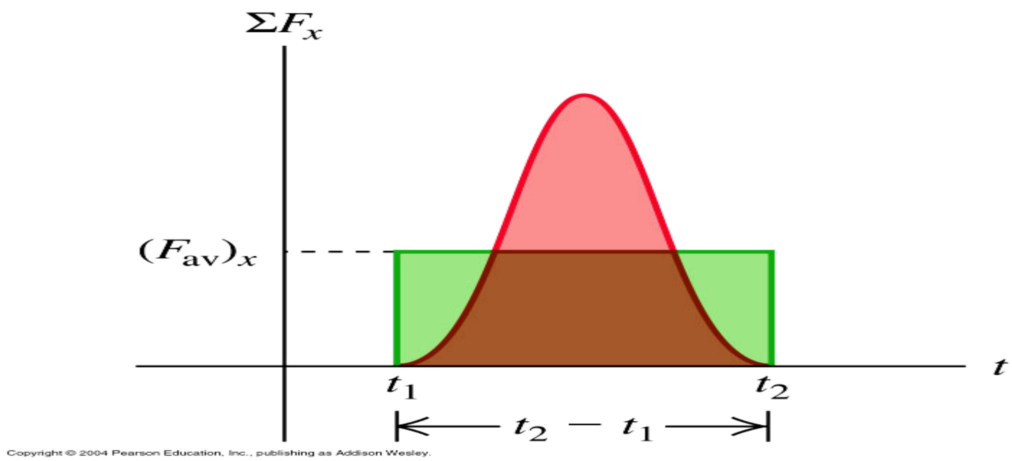
<u>The 2nd Rocket Equation</u>: $$V_f - V_i = V_{rel} (\ln \frac{M_i}{M_f})$$

Where M_i **is the rockets initial mass and** M_f **its mass after the fuel burn.**

Collisions

When two or more objects collide, there are brief relatively large forces that require an averaging over time in order to quantify.

During the collision, both the energy and linear momentum of the colliding objects may be conserved quantities.



An analytic form of the collision force is often unknown. The exact area under this curve is $A = \int_{t_i}^{t_f} F(t)dt$, but in the absence of an exact knowledge of $\vec{F}(t)$ the approximation $F_{Av}\Delta t$ is convenient.

The integral definition of collision **impulse** is: $\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)dt$

Which using our approximation gives $|\vec{J}| = F_{Avg}\Delta t$

Where Δt is the collision duration.

From the 2nd law $\vec{F} = \frac{\Delta \vec{P}}{\Delta t}$

$$|\vec{J}| = F_{Avg} \Delta t$$

$$\vec{J} = \overrightarrow{P_f} - \overrightarrow{P_i} = \Delta \vec{P}$$

The change in linear momentum is equal to the collision impulse.

Since impulse is a vector quantity, we can write the equation in each coordinate:

$$J_x = P_{f,x} - P_{i,x}$$

$$J_y = P_{f,y} - P_{i,y}$$

$$J_z = P_{f,z} - P_{i,z}$$

E.g. Take a 0.14 kg baseball pitched with an initial velocity of V_i = -39 m/s (toward the batter) and hit back toward the field at a 30 degree upward angle with a final velocity of V_f = 45 m/s. Collision time is 1.2 ms.

$$J_x = P_{f,x} - P_{i,x} = MV_{f,x} - MV_{i,x} = 10.92\_kgm/s$$

$$J_y = P_{f,y} - P_{i,y} = MV_{f,y} - MV_{i,y} = 3.15\_kgm/s$$

The collision impulse is $\vec{J} = (10.92\hat{i} + 3.15\hat{j})\_kgm/s$

The magnitude is $|\vec{J}| = \sqrt{(10.92)^2 + (3.15)^2} = 11.4\_kgm/s$

The direction is $\theta = Tan^{-1}\frac{3.15}{10.92} = 16^0$

$$F_{Avg} = \frac{|\vec{J}|}{\Delta t} = \frac{11.4\_kgm/s}{1.2x10^{-3}s} \sim 9500N$$

Notice that $mg = 1.37N$ due to gravity is negligible in comparison with

$F_{Avg} \sim 9500N$ such that this system is effectively **isolated**.

Impact of several particles of mass M and speed V on a fixed target:

For n objects incident on the target, each mass M receives an impulse $J = \Delta P$

Target impulse over time Δt is therefore: $J = -n\Delta P$

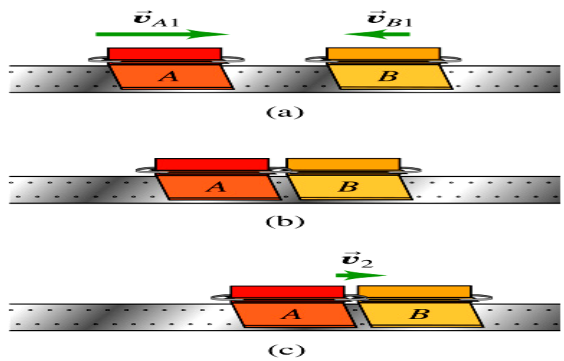
The corresponding force on the target:

$$F_{Avg} = \frac{|\vec{J}|}{\Delta t} = \frac{-n\Delta P}{\Delta t} = \frac{-nM\Delta V}{\Delta t}$$

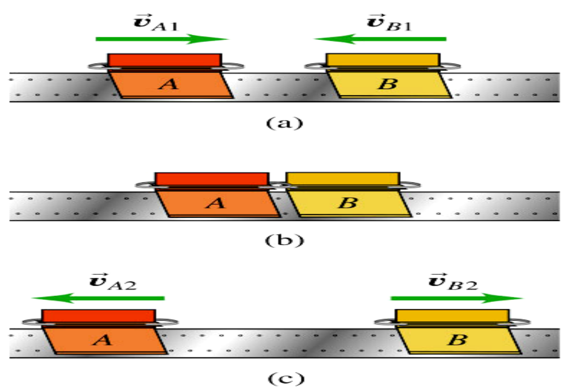
$\frac{n}{\Delta t}$ Is the particle rate. The force on the target is therefore proportional to incident rate.

Collision Types:

Particles colliding and sticking to the target are **[completely inelastic collisions](#)**.



If particles rebound without deformation from the target then this is an **[elastic collision](#)**



Hail impacts with twice the force of rain since $\Delta\vec{P}$ is twice as large for elastic collisions.

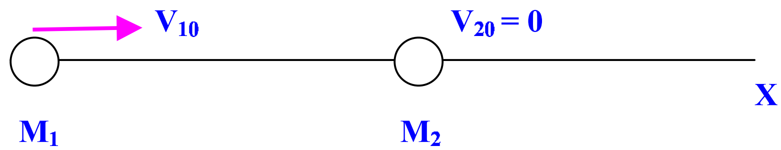
When the incident colliding particle sticks a little but eventually separates from the target, the collision is termed **[partially inelastic](#)**.

Elastic collisions conserve both linear momentum and kinetic energy

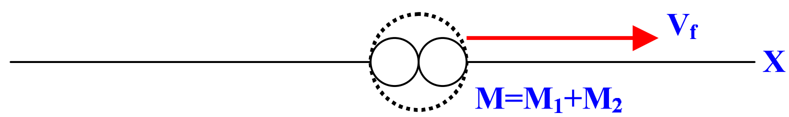
Completely and partially inelastic collisions only conserve linear momentum.

Completely Inelastic Collisions in One-Dimension:

Before:



After:



Conserving linear momentum:

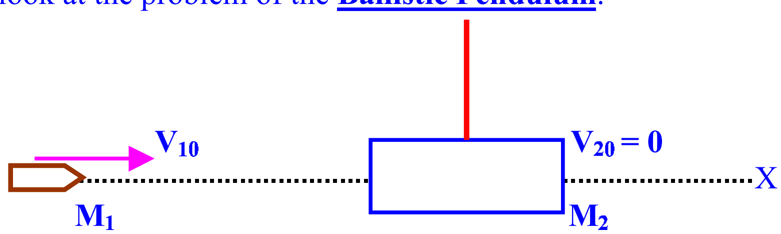
$$P_{iX} = P_{fX}$$

$$M_1 V_{10} + M_2 V_{20} = (M_1 + M_2) V_f$$

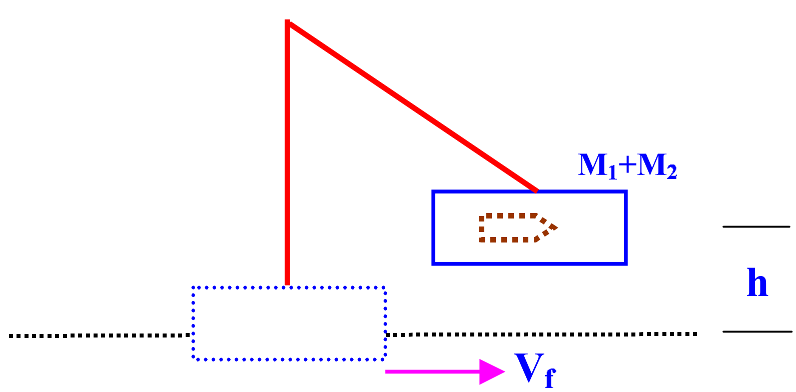
$$V_f = \frac{M_1 V_{10}}{M_1 + M_2}$$

Using this result, we can look at the problem of the **Ballistic Pendulum**:

Before:



After:



Conserving Momentum,

$$P_{iX} = P_{fX}$$

$$M_1 V_{10} = (M_1 + M_2) V_f$$

$$V_{10} = \frac{(M_1 + M_2) * V_f}{M_1}$$

Conserving energy **after** the collision:

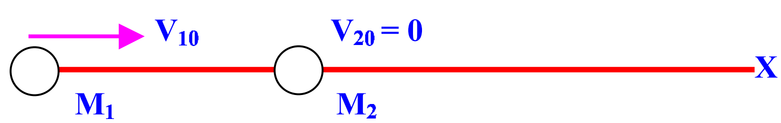
$$\frac{1}{2}(M_1 + M_2) * V_f^{\ 2} = (M_1 + M_2) g h$$

$$V_f = \sqrt{2gh}$$

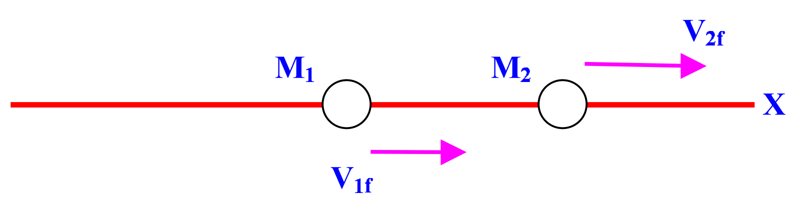
$$V_{10} = \frac{(M_1 + M_2) * \sqrt{2gh}}{M_1}$$

Elastic Collision in 1-Dimension

Before:



After:



Conserving linear momentum in the coordinate X:

$$M_1 V_{10} = M_1 V_{1f} + M_2 V_{2f}$$

Conserving kinetic energy:

$$\frac{1}{2} M_1 * V_{10}{}^2 = \frac{1}{2} M_1 V_{1f}{}^2 + \frac{1}{2} M_2 V_{2f}{}^2$$

From the momentum conservation:

$$V_{1f} = \frac{M_1 V_{10} - M_2 V_{2f}}{M_1}$$

Which we substitute:

$$\frac{1}{2} M_1 * V_{10}{}^2 = \frac{1}{2} M_1 \{\frac{M_1 V_{10} - M_2 V_{2f}}{M_1}\}^2 + \frac{1}{2} M_2 V_{2f}{}^2$$

Rearranging:

$$\frac{1}{2} M_1 * V_{10}{}^2 = \frac{1}{2M_1} (M_1 V_{10} - M_2 V_{2f})^2 + \frac{1}{2} M_2 V_{2f}{}^2$$

$$\frac{1}{2} M_1 * V_{10}{}^2 = \frac{1}{2M_1} (M_1{}^2 V_{10}{}^2 - 2M_1 M_2 V_{10} V_{2f} + M_2{}^2 V_{2f}{}^2) + \frac{1}{2} M_2 V_{2f}{}^2$$

$$0 = -M_2 V_{10} V_{2f} + \frac{1}{2M_1} M_2{}^2 V_{2f}{}^2 + \frac{1}{2} M_2 V_{2f}{}^2$$

Multiplying left and right hand sides by $\frac{2}{M_2 V_{2f}}$

$$0 = -2V_{10} + \frac{M_2}{M_1} V_{2f} + V_{2f}$$

$$2V_{10} = V_{2f}\left[1 + \frac{M_2}{M_1}\right]$$

Solving for V_{2f}

$$V_{2f} = \frac{2V_{10}}{1 + \frac{M_2}{M_1}} = \frac{2M_1}{M_1 + M_2} V_{10}$$

From this V_{1f} **follows:**

$$V_{1f} = \frac{M_1 V_{10} - M_2 V_{2f}}{M_1} = V_{10} - \frac{2V_{10}}{1 + \frac{M_1}{M_2}}$$

$$V_{1f} = V_{10} * \left[\frac{M_1 - M_2}{M_1 + M_2}\right]$$

Looking at three particular cases, we can examine the limiting behavior for the process:

1) **$M_1 = M_2$** **Identical mass for incident and target particle**

2) **$M_1 << M_2$** **Massive target**

3) **$M_1 >> M_2$** **Massive projectile**

For case 1:	For case 2:	For case 3:
$V_{2f} = V_{10}$ $V_{1f} = 0$	$V_{2f} = \frac{2M_1}{M_2} V_{10}$ $V_{1f} = -V_{10}$	$V_{2f} = 2V_{10}$ $V_{1f} = V_{10}$

COM velocity before and after the collision

Since the external forces are effectively zero during the collision, the COM acceleration is zero and therefore the COM velocity constant:

Before the collision:

$$V_{COM} = \frac{\sum_i M_i V_i}{\sum_i M_i} = \frac{M_1 V_{10} + M_2 V_{20}}{M_1 + M_2} = \frac{M_1 V_{10}}{M_1 + M_2}$$ Mass 2 initially at rest.

After the collision:

$$V_{COM} = \frac{\sum_i M_i V_i}{\sum_i M_i} = \frac{M_1 V_{1f} + M_2 V_{2f}}{M_1 + M_2} = \frac{M_1 V_{10} - \dfrac{2M_1 V_{10}}{1 + \dfrac{M_1}{M_2}} + M_2 \dfrac{2V_{10}}{1 + \dfrac{M_2}{M_1}}}{M_1 + M_2}$$

$$V_{COM} = \frac{M_1 V_{10}}{M_1 + M_2}$$

General One-Dimensional Elastic Collision:

Conserving linear momentum in the coordinate X:

$$M_1 V_{10} + M_2 V_{20} = M_1 V_{1f} + M_2 V_{2f}$$

Conserving kinetic energy:

$$\frac{1}{2} M_1 * V_{10}{}^2 + \frac{1}{2} M_2 * V_{20}{}^2 = \frac{1}{2} M_1 V_{1f}{}^2 + \frac{1}{2} M_2 V_{2f}{}^2$$

2 Equations with 2 unknowns solves:

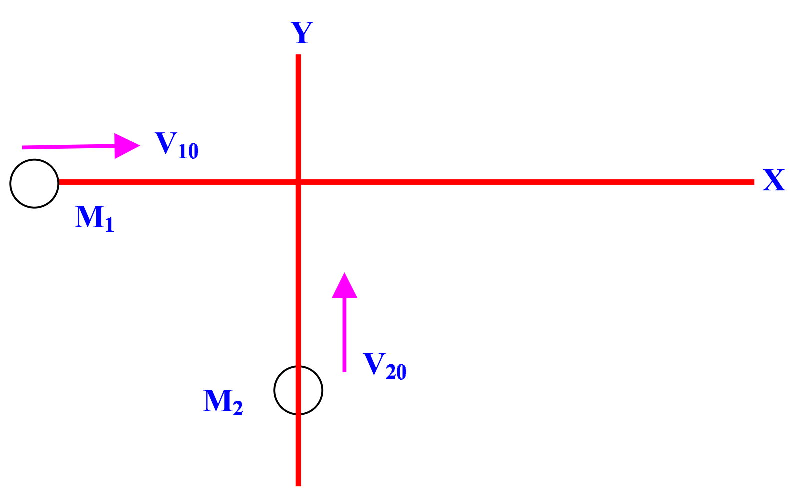
$$V_{1f} = V_{10} * \left[\frac{M_1 - M_2}{M_1 + M_2} \right] + V_{20} * \left[\frac{2M_2}{M_1 + M_2} \right]$$

$$V_{2f} = V_{20} * \left[\frac{M_2 - M_1}{M_1 + M_2} \right] + V_{10} * \left[\frac{2M_1}{M_1 + M_2} \right]$$

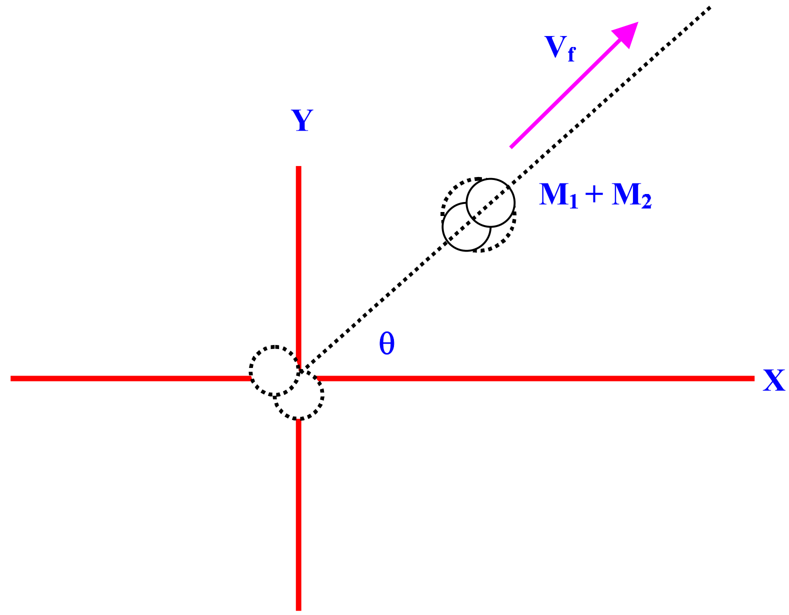
Notice the $1 \leftrightarrow 2$ symmetry in these results; motion is relative.

Two-Dimensional Completely Inelastic Collision:

Before:



After:



Conserving linear momentum in the coordinate X:

$$P_{iX} = P_{fX} \qquad M_1 V_{10} = (M_1 + M_2)V_{fX}$$

Conserving linear momentum in the coordinate Y:

$$P_{iY} = P_{fY} \qquad M_2 V_{20} = (M_1 + M_2)V_{fY}$$

$$V_{fX} = V_f Cos(\theta) \qquad V_{fY} = V_f Sin(\theta)$$

The two momentum equations give:

$$Tan(\theta) = \frac{M_2 V_{20}}{M_1 V_{10}} \qquad V_{fx} = \frac{M_1 V_{10}}{M_1 + M_2} \qquad V_{fy} = \frac{M_2 V_{20}}{M_1 + M_2}$$

$$V_f = \sqrt{V_{fx}^{\ 2} + V_{fy}^{\ 2}} = \frac{1}{M_1 + M_2}\left[\sqrt{(M_1 V_{10})^2 + (M_2 V_{20})^2}\right]$$