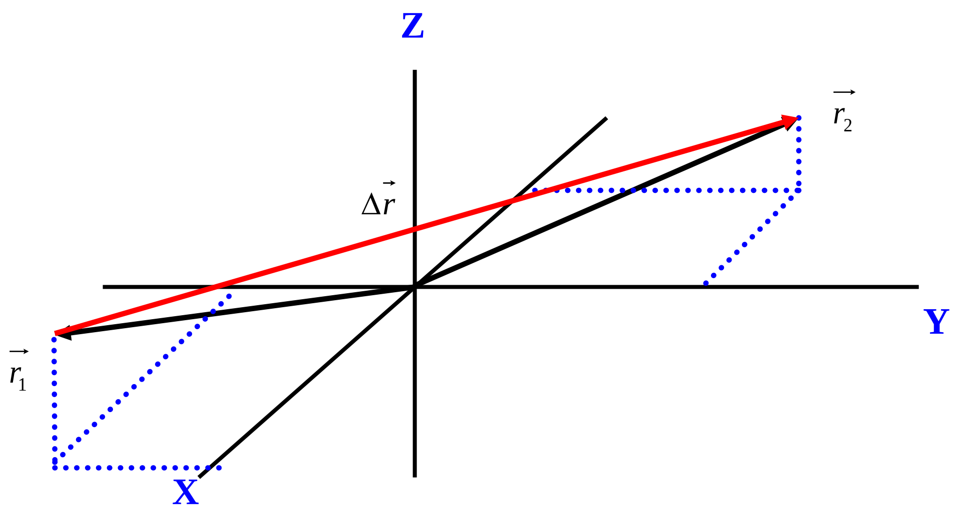
Motion in Two or Three Dimensions:

The notation for particle position in a 3-D coordinate frame is: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

For example, starting out at position one as $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and moving to position two $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, we write the displacement $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$ as:

$$\Delta\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$



From definitions for average velocity and acceleration:

$$\vec{V}_{avg} = \frac{\Delta\vec{r}}{\Delta t} \qquad \vec{a}_{avg} = \frac{\Delta\vec{V}}{\Delta t}$$

The instantaneous velocity and acceleration are:

$$\vec{V}(t) = \frac{d\vec{r}}{dt}$$

$\vec{V}(t)$ **Is tangent to the particle path**.

$$\vec{a}(t) = \frac{d\vec{V}}{dt}$$

Projectile Motion:

In a simplified two-dimensional treatment of projectile motion, we assume that a_x is identically zero, i.e., there is no acceleration in the x-coordinate motion. Y-coordinate motion corresponds to free fall motion.

Zero acceleration in the x-coordinate motion is an over simplification since there are drag forces and non-inertial reference frame effects, for example which change the idealized trajectories we derive here.

Writing down the kinematics equations for a projectile in an **x-y** plane with the usual convention that the 'y' axis is the vertical direction:

With $a_x = 0$, in the x-coordinate:

Kinematics Equation 1: $v_x = v_{0x}$ **Unchanged in the motion**

Kinematics Equation 2: $x = x_0 + v_{0x}t$ **Setting** $x_0 = 0$ **this is** $x = v_{0x}t$

Kinematics Equation 3: $v_x{}^2 = v_{0x}{}^2$ **Nothing new here, same as #1**

With $a_y = -g$:

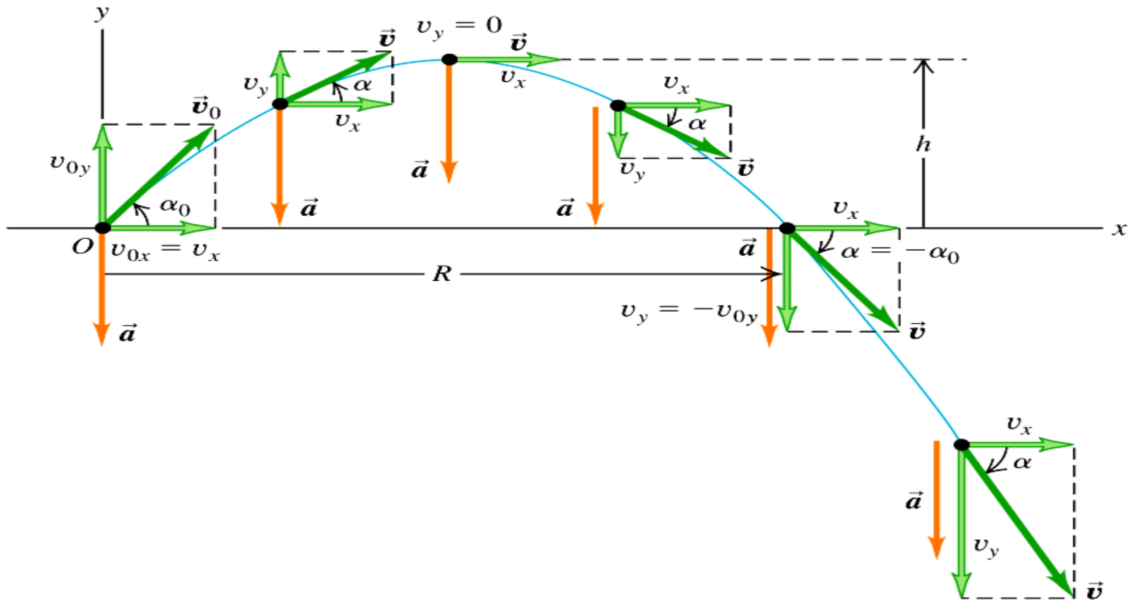
Kinematics Equation 1: $v_y = v_{0y} - gt$

Kinematics Equation 2: $y = y_0 + v_{0y}t - \frac{1}{2}gt^2$

Kinematics Equation 3: $v_y{}^2 = v_{0y}{}^2 - 2g\Delta y$

Trajectory Equation:

Taking $x_0 = 0$ and an initial projectile velocity vector of magnitude $|\vec{v}_0|$ at angle α_0



$$v_{0x} = v_0 Cos(\alpha_0)$$

$$v_{0y} = v_0 Sin(\alpha_0)$$

Using x-Kinematics Equation 2

$$x = v_{0x} t$$

$$t = \frac{x}{v_0 Cos(\alpha_0)}$$

Using y-Kinematics Equation 2

$$y = y_0 + v_{0y} t - \frac{1}{2} g t^2 = y_0 + v_0 Sin(\alpha_0) \cdot \frac{x}{v_0 Cos(\alpha_0)} - \frac{1}{2} g \left(\frac{x}{v_0 Cos(\alpha_0)} \right)^2$$

$$y = y_0 + Tan(\alpha_0) \cdot x - \frac{g}{2v_0^{\,2} Cos^2(\alpha_0)} \cdot x^2$$

Parabolic trajectory.

The Range Formula:

By setting $y_0 = 0$ in the equation $y = y_0 + v_{0y}t - \frac{1}{2}gt^2$,

We can solve for the times for which the condition $y = 0$ is true:

$$0 = v_{0y}t - \frac{1}{2}gt^2$$ Giving $$t = 0$$ **or** $$v_{0y} - \frac{1}{2}gt = 0$$

$t = 0$ **Corresponding to the launch point.**

$t = \frac{2v_{0y}}{g}$ **Corresponding to the landing point.**

At the time $t = \frac{2v_{0y}}{g}$ we solve for x:

$$x = v_{0x}t = v_0 Cos(\alpha_0) \cdot \frac{2v_{0y}}{g} = \frac{2v_0^{\,2} Sin(\alpha_0) Cos(\alpha_0)}{g}$$

$$x = \frac{v_0^{\,2} Sin(2\alpha_0)}{g}$$

Note three features here:

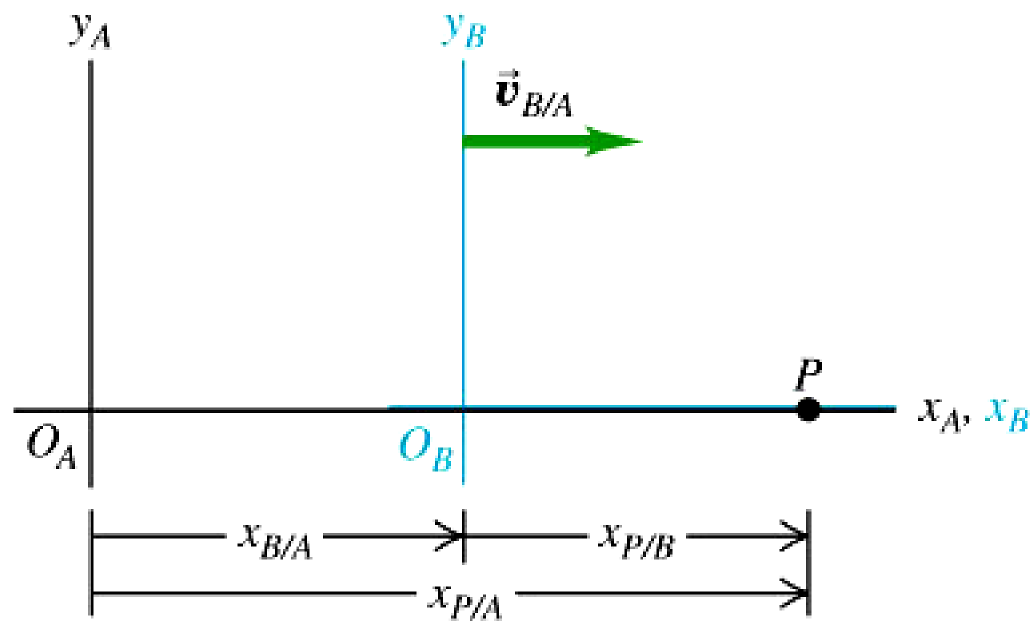
1) $y = y_0 = 0$ Derivation proceeded under the condition that the takeoff height above ground was equal to the landing height above ground.

2) x Is a maximum for $\alpha_0 = 45^0$

3) There are two angles, one above $\alpha_0 = 45^0$ and one below $\alpha_0 = 45^0$ for which the horizontal distance values are identical.

Relative Motion:

An object is moving in a frame of reference where that frame is itself in motion with a **constant velocity**. We wish to evaluate the object kinematics from the perspective of a reference frame that is stationary with respect to the first. By relating the two frames relative motion, we may determine the object motion from the 'at-rest' point of view.



(b)

In one dimension the position X is:

$X_{P/A} = X_{P/B} + V_{B/A} * t$ Where $V_{B/A}$ is the velocity of the frame **B wrt frame A**.

Evaluating the time rate of change on both sides of this equation:

$$V_{P/A} = V_{P/B} + V_{B/A}$$

In general for two or three dimensions these relations are vector equations so that:

$$X_{P/A} = X_{P/B} + V_{B/A_X} * t$$
$$Y_{P/A} = Y_{P/B} + V_{B/A_Y} * t$$
$$Z_{P/A} = Z_{P/B} + V_{B/A_Z} * t$$

$V_{rest} = V_{moving} + V$ **[Vel. in rest frame = Vel. in moving + Vel. of moving frame]**

Consider a boat in motion relative to the 'stationary' Earth at V_{be} and moving in a river with currents at speed V_{re}

Mnemonically $\vec{V}_{be} = \vec{V}_{br} + \vec{V}_{re}$

That is, V **(Boat wrt Earth) =** V **(Boat wrt River) +** V **(River wrt Earth).**

Notice that in the equation $V_{rest} = V_{moving} + V$ since V is a constant if we take the time rate of change on both sides, the change in V is equal to zero and we are left with:

$$a_{rest} = a_{moving}$$

Since Newton's Second Law is $F = ma$, both frames of reference see the same forces on the object. The primed frame is an **inertial reference frame**.