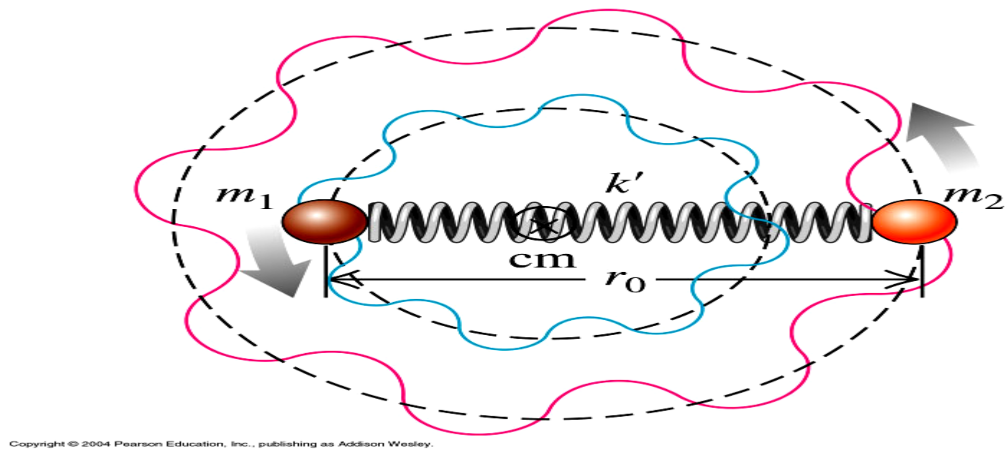
Oscillations

Description of matter at the molecular level views atoms as bound oscillating masses. Atoms modeled as **mass-spring oscillating mechanical systems** leads to theoretical predictions that are correct experimentally.



Macroscopic oscillations as in wave motions, pendulums, oscillating circuits…etc. also indicate the importance of understanding vibrations and oscillatory motion.

Simple Harmonic Motion

An object whose position is **periodic** or repeats in regular time intervals, and is a sinusoidal function of time is referred to as executing **simple harmonic motion**.

$$x(t) = x_m Cos(\omega t + \phi)$$

x_m is **amplitude** and is equal to the maximum +/- displacement of the mass location.

$\omega = 2\pi / T$ is **angular frequency** of the oscillator.

ϕ is a **phase factor or phase angle** in units of radians.

f is **frequency** or number of oscillations per second. Units of f are Hertz Hz .

Since the motion is periodic,

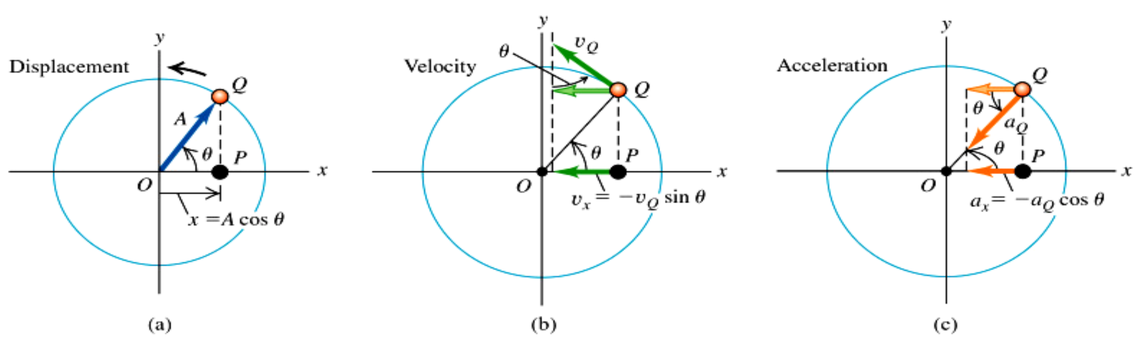
$$x_m Cos(\omega t) = x_m Cos[\omega(t+T)] \quad \rightarrow \quad \omega T = 2\pi$$

$$\omega = 2\pi / T \qquad\qquad \omega = 2\pi f$$

Velocity and Acceleration

Given a position function, velocity and acceleration may be obtained by evaluating derivative or variations in time of the position function

$$v(t) = -\omega x_m Sin[\omega t + \varphi]$$
$$a(t) = -\omega^2 x_m Cos[\omega t + \varphi] = -\omega^2 x(t)$$



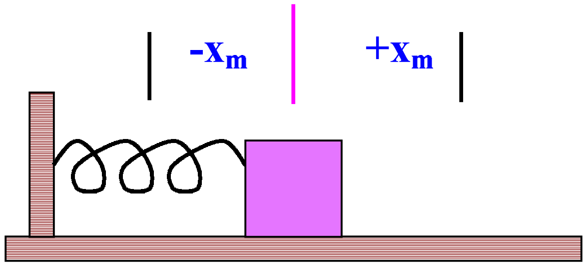
The **velocity amplitude** is ωx_m

The **acceleration amplitude** is $\omega^2 x_m$

1) **The displacement maxima +/- correspond to velocity minima.**
2) **The displacement maxima correspond to acceleration maxima.**
3) **There is a relative minus sign between x(t) and a(t).**

Linear Simple Harmonic Oscillator

The **restoring force** leading to oscillations about equilibrium obeys **Hooke's Law** for springs in the elastic region: $F = -kx$



Newton's 2nd Law is:

$$F = -kx = ma$$

$$a = -\omega^2 x$$

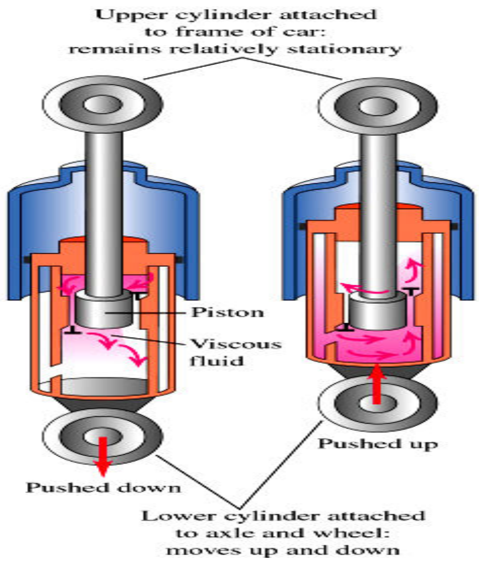
$$-kx = -m\omega^2 x$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

Stiffer springs (large k) give systems smaller T

Oscillating systems may also have **damping** and/or be driven by external **driving forces**.



Mechanical Energy

The mechanical energy of an **un-damped** oscillator is the sum of kinetic and spring potential energies:

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}m[-\omega x_m Sin(\omega t + \phi)]^2$$

$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}k[x_m Cos(\omega t + \phi)]^2$$

$$E = K(t) + U(t) = \frac{1}{2}kx_m{}^2\{Sin^2(\omega t + \phi) + Cos^2(\omega t + \phi)\} = \frac{1}{2}kx_m{}^2$$

The mechanical energy is a constant provided damping is absent.

Oscillator motion is alternately maximum kinetic energy when the spring has zero extension or compression and maximum potential energy at the **turning points** where the mass velocity is zero.

Solving for the ***velocity of the mass at any position,***

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_m{}^2 \qquad \frac{1}{2}mv^2 = \frac{1}{2}kx_m{}^2 - \frac{1}{2}kx^2$$

$$v^2 = \frac{k}{m}x_m{}^2 - \frac{k}{m}x^2$$

$$v = \pm\sqrt{\frac{k}{m}} * x_m\sqrt{1 - \frac{x^2}{x_m{}^2}} = \pm\omega * x_m\sqrt{1 - \frac{x^2}{x_m{}^2}}$$

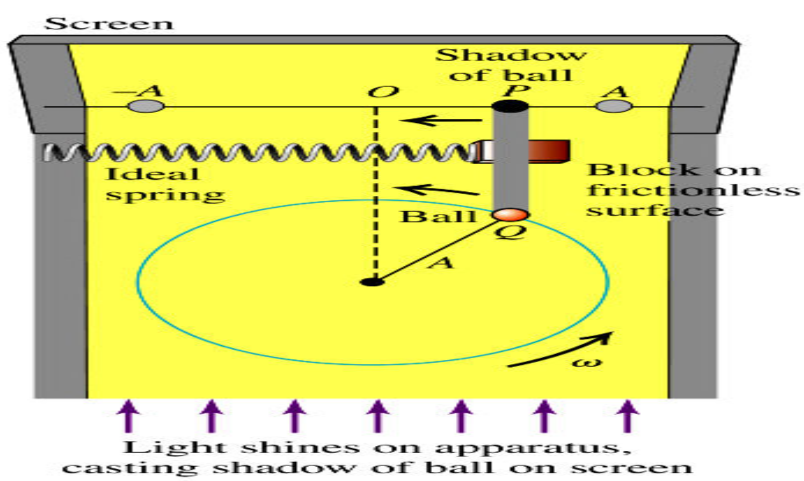
Simple Harmonic Motion and Uniform Circular Motion

If an object is in uniform circular motion at radius x_m, then given an initial condition $x(t=0) = x_m$, the position function is:

$$x(t) = x_m Cos(\omega t)$$

ω is the angular frequency of UCM

Projecting motion onto the X-axis gives a **reference particle** that executes simple harmonic motion between **[-A ,+A]:**



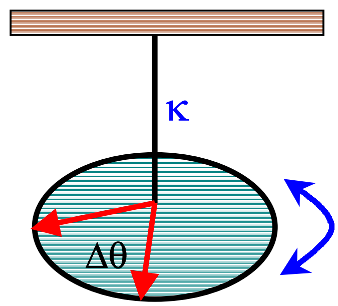
Driving a piston in simple harmonic motion can therefore lead to rotational motion.

Pendulums

Pendulums also exhibit simple harmonic motion for **small angular displacements** of the pendulum arm. Consider here three types of pendulums:

1) **Torsion Pendulum**
2) **Simple Pendulum**
3) **Physical Pendulum**

The torsion pendulum consists of a horizontally oriented disk suspended by a wire with **torsion constant κ.**



As the disk is rotated from equilibrium a restoring torque from the wire results:

$$\tau = -\kappa\theta$$

Newton's 2nd Law is then: $I\alpha = -\kappa\theta$

$$\theta(t) = \theta_m Cos(\omega t)$$

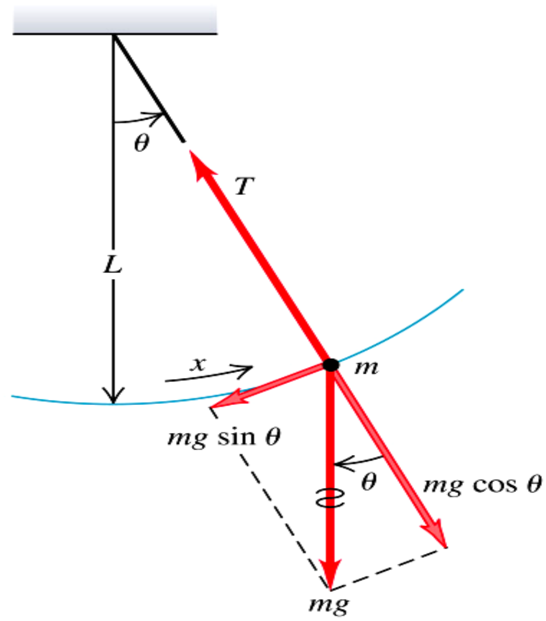
$$\Omega(t) = -\omega\theta_m Sin(\omega t) \qquad \alpha(t) = -\omega^2\theta_m Cos(\omega t)$$

$$\omega = \sqrt{\frac{\kappa}{I}}$$

Since κ is dependent on wire length, diameter and material type, the period

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{\kappa}}$ may be adjusted by changing any of these quantities.

Simple Pendulum:



$$\sum \tau = I\alpha \qquad\qquad -LmgSin(\theta) = I\alpha$$

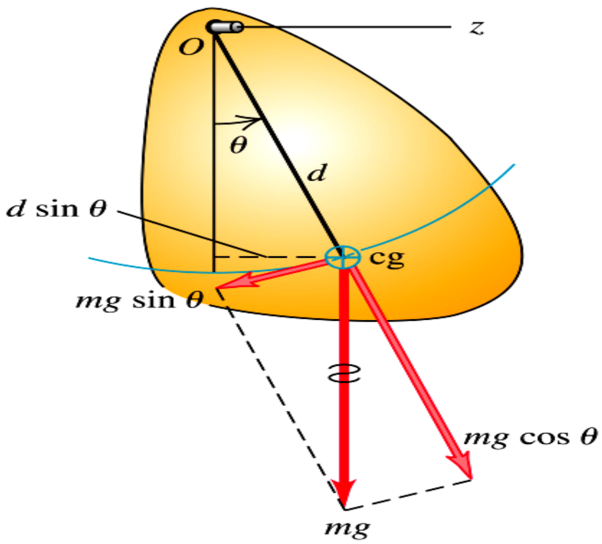
Small angle approximation ➔ $Sin(\theta) \cong \theta$

Solving as before $\omega = \sqrt{\dfrac{mgL}{I}}$

For a point mass on the pendulum $I = mL^2$

And $\omega = \sqrt{\dfrac{g}{L}}$ **Independent of mass**.

With a physical pendulum, I can be arbitrary and $\omega = \sqrt{\dfrac{mgd}{I}}$ where d is the distance from the axis of rotation to the pendulum center of gravity.



When $d = 0$ and the axis of rotation is through the object **COG**, the angular frequency **is zero** and oscillation does not take place.

The **center of oscillation** is the distance from the axis of rotation for which the period of

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L_0}{g}}$$ equals the period of a physical pendulum $$2\pi\sqrt{\frac{I}{mgd}}$$

Equating the two periods leads to $L_0 = \dfrac{I}{md}$

Damped Simple Harmonic Motion

Most oscillating systems are subject to damping that halts oscillatory motion after some amount of time. For damping forces proportional to object velocity, Newton's 2nd law is:

$$\sum F = ma$$

$-kx - bv = ma$ Here b is the damping constant.

Setting $2\gamma = b/m$ and since $\omega_0^{\;2} = k/m$

$x(t) = Ae^{(-\gamma+\sqrt{\gamma^2-\omega_0^{\;2}})*t}$ Or $x(t) = Be^{(-\gamma-\sqrt{\gamma^2-\omega_0^{\;2}})*t}$

The general solution is a linear combination of the two possibilities:

$$x(t) = e^{-\gamma t}\{Ae^{(\sqrt{\gamma^2-\omega_0^{\;2}})*t} + Be^{-(\sqrt{\gamma^2-\omega_0^{\;2}})*t}\}$$

Three cases exist depending on whether $\gamma^2 < \omega_0^{\;2}$, $\gamma^2 > \omega_0^{\;2}$, or $\gamma^2 = \omega_0^{\;2}$

Case 1 Underdamped Solution:

$\gamma^2 < \omega_0^{\;2}$ ➔ $b < 2\sqrt{\dfrac{k}{m}}$

$$x(t) = e^{-\gamma t}\{Ae^{i(\sqrt{\omega_0^{\;2}-\gamma^2})*t} + Be^{-i(\sqrt{\omega_0^{\;2}-\gamma^2})*t}\}$$

$$x(t) = e^{-\gamma t}\{(A+B)Cos(\sqrt{\omega_0^{\;2}-\gamma^2})t + (A-B)iSin(\sqrt{\omega_0^{\;2}-\gamma^2})t\}$$

$$x(t) = e^{-\gamma t}\{A'Cos(\omega' t) + B'Sin(\omega' t)\}$$

With initial conditions $x(0) = x_m$ and $v(0) = 0$,

$A' = x_m$

$B'\omega' - A'\gamma = 0$

$B' = A'\gamma/\omega' = x_m \{\gamma/\omega'\}$

$$x(t) = x_m e^{-\gamma t}\{Cos(\omega' t) + \frac{\gamma}{\omega'}Sin(\omega' t)\}$$

Wishing to write something like:

$$x(t) = x_m \frac{\omega_0}{\omega'} e^{-\gamma t}\{Cos(\omega' t + \phi)\}$$

The phase angle ϕ must be:

$$\phi = -Tan^{-1}\frac{\gamma}{\omega'}$$

Proving this is so we must show:

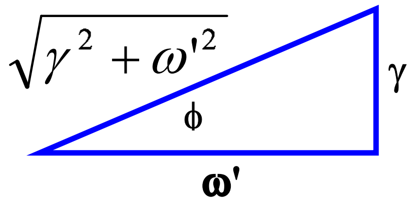
$$\frac{\omega_0}{\omega'}\{Cos(\omega' t + \phi)\} = Cos(\omega' t) + \frac{\gamma}{\omega'}Sin(\omega' t)$$

From trigonometry the LHS is

$$\frac{\omega_0}{\omega'}\{Cos(\omega' t)Cos(\phi) - Sin(\omega' t)Sin(\phi)\}$$

$$Cos(\phi) = Cos\{Tan^{-1}\frac{\gamma}{\omega'}\}$$

The reference triangle is:



Now $\sqrt{\gamma^2 + \omega'^2}$ is equal to ω_0

Such that $$Cos(\phi) = \omega' / \omega_0$$

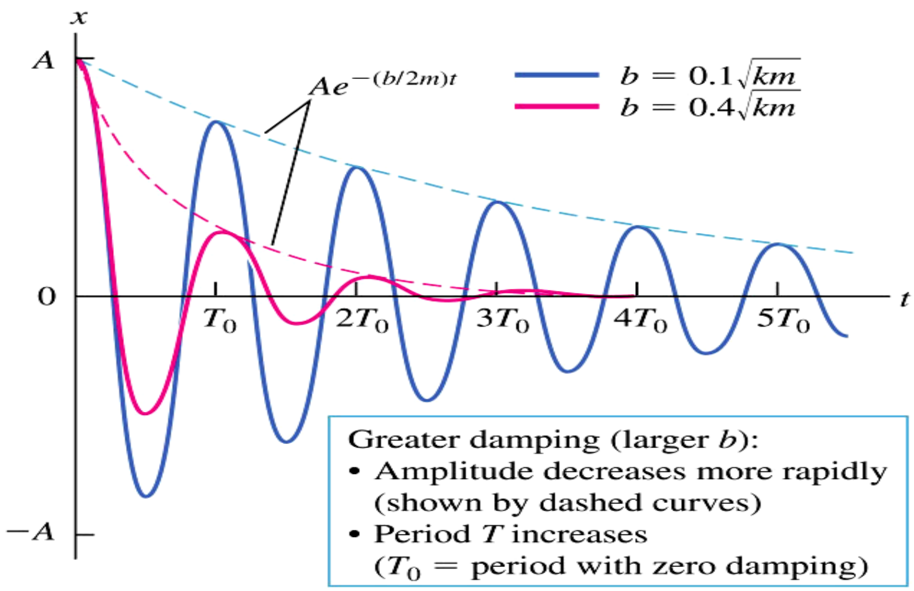
And $$Sin(\phi) = -\gamma / \omega_0$$

$$\frac{\omega_0}{\omega'}\{Cos(\omega' t + \phi)\} = Cos(\omega' t) + \frac{\gamma}{\omega'} Sin(\omega' t)$$

$$x(t) = x_m \frac{\omega_0}{\omega'} e^{-\gamma t}\{Cos(\omega' t + \phi)\}$$

Where $\phi = -Tan^{-1}\frac{\gamma}{\omega'}$ $\qquad$ $\omega' = \sqrt{\omega_0{}^2 - \gamma^2}$

The underdamped case gives an oscillation that decays exponentially with time.



Critically Damped Case:

$$\gamma^2 = \omega_0{}^2 \quad \rightarrow \quad b = 2\sqrt{\frac{k}{m}}$$

In this case ω' is identically zero and we can evaluate the limiting form of:

$$x(t) = x_m e^{-\gamma t}\{1 + \gamma t\}$$

Mass returns to equilibrium as quickly as possible without oscillation about equilibrium.

Case 3; Overdamped Case:

$$\gamma^2 > \omega_0{}^2 \quad \Rightarrow \quad b > 2\sqrt{\frac{k}{m}}$$

Starting with $$x(t) = x_m e^{-\gamma t}\{Cos(\omega' t) + \frac{\gamma}{\omega'} Sin(\omega' t)\}$$

$$\omega' = \sqrt{\omega_0{}^2 - \gamma^2} = i\sqrt{\gamma^2 - \omega_0{}^2}$$

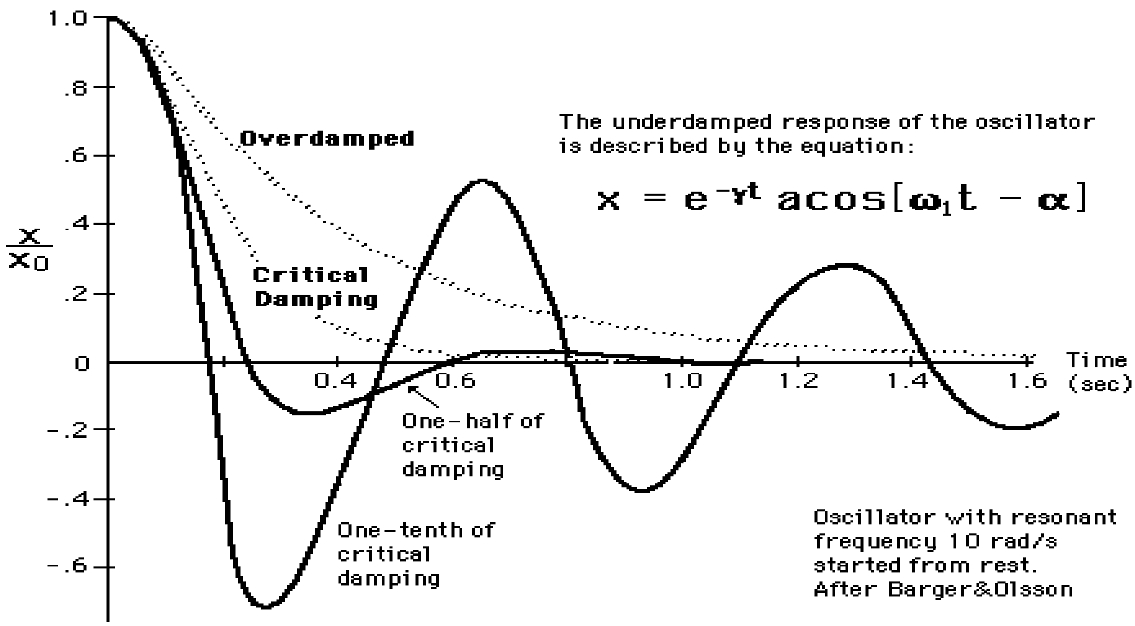
Now $$Cos(ix) = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{x} + e^{-x}}{2} = Cosh(x)$$

And $$Sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2i} = iSinh(x)$$

$$x(t) = x_m e^{-\gamma t}\{Cosh(\sqrt{\gamma^2 - \omega_0{}^2}\, t) + \frac{\gamma}{\omega'} Sinh(\sqrt{\gamma^2 - \omega_0{}^2}\, t)\}$$

$$x(t) = x_m \frac{\omega_0}{\omega'} e^{-\gamma t}\{Cosh(\omega' t + \phi)\}$$

$$\omega' = \sqrt{\gamma^2 - \omega_0{}^2}$$



Resonance:

Given a driving force that is also simple harmonic, it becomes possible to drive an oscillator system into a **resonance condition** where large amplitude oscillations eventually lead to mechanical breakdown of the system. The condition for resonance is:

$$\omega_{Driving} = \omega_0$$

(In reality, some damping makes this approximately true)

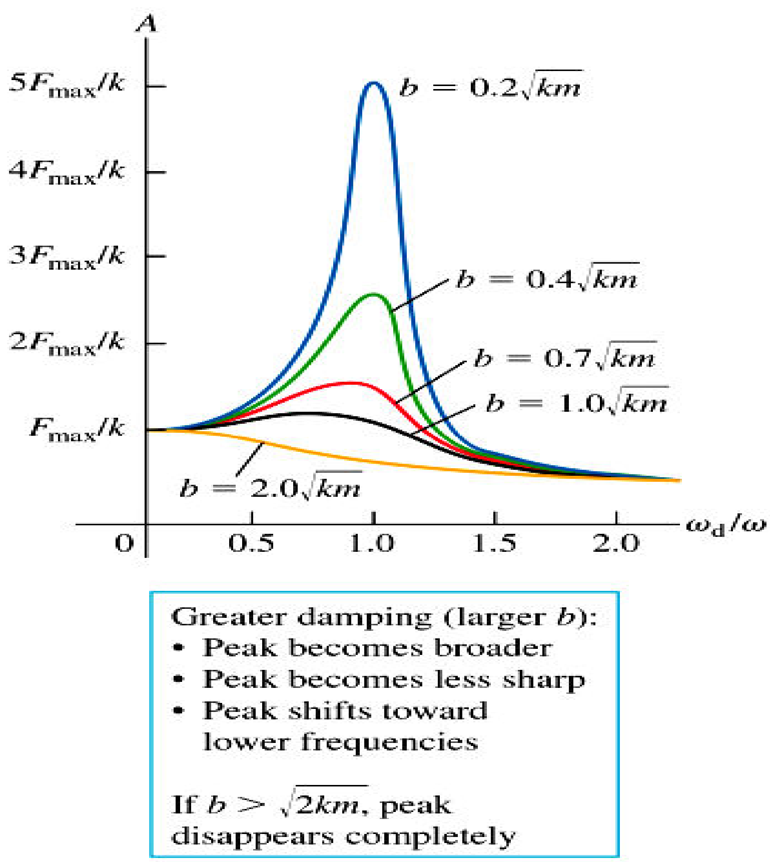
The amplitude when driving an ideal system at resonance is:

$$x_m^{\ 2} = \frac{(F/m)^2}{\omega_0^{\ 2} - \omega_{Driving}^{\ 2}}$$

By damping the system resonance at the natural frequency is avoided:

$$x_m^{\ 2} = \frac{(F/m)^2}{\omega_0^{\ 2} - \omega_{Driving}^{\ \ 2} + \dfrac{\omega_{Driving}^{\ \ 2} b^2}{m^2}}$$

Resonant amplitude peaks become larger and have narrower half-width, which equals **b**, as damping is reduced.

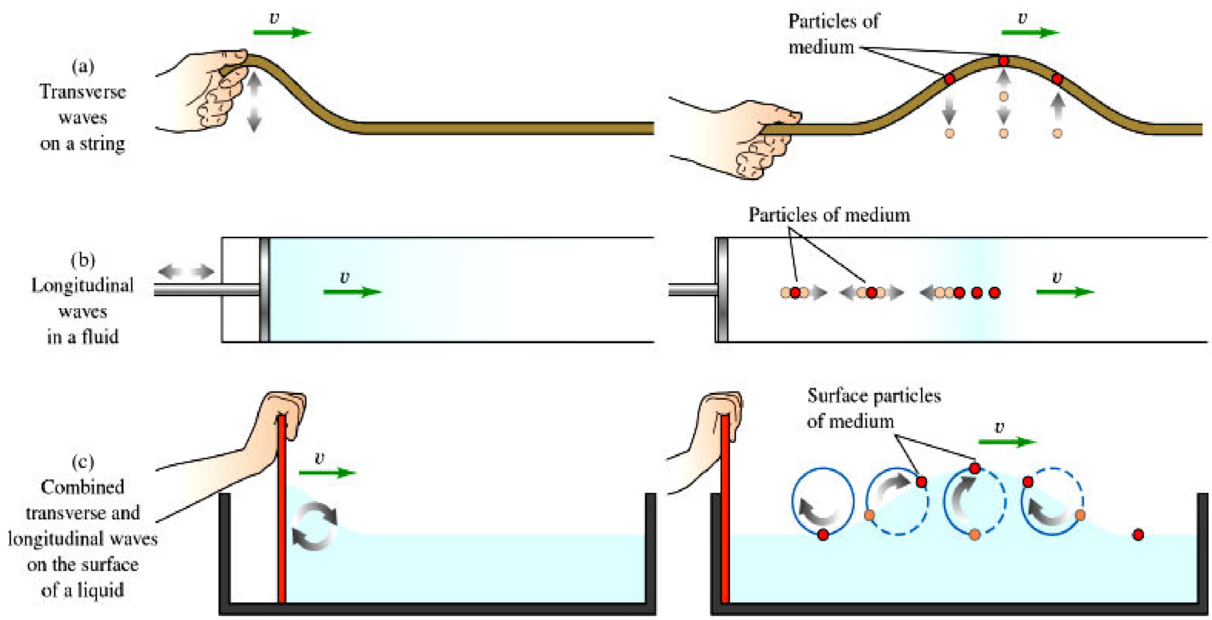


Finally, it is not always the case that resonance is something to be avoided. Many electronics circuits 'tune' in on particular frequencies by having components that when combined will resonate at those frequencies. We introduce these circuits in Physics II.

Wave Motion

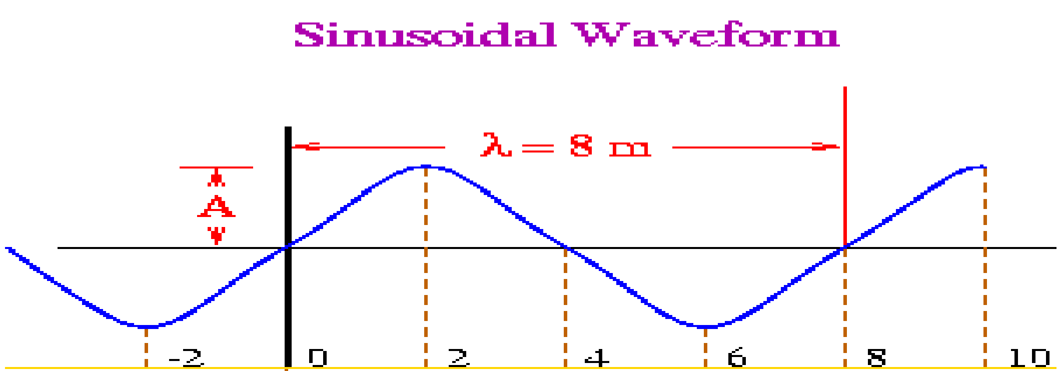
Mechanical waves like sound waves, waves along a rope or waves in a pond provide examples of oscillation phenomena on a macroscopic level.

The wave consists of oscillations that displace the medium through which it travels, but the wave **does not transport matter** from one point to another. **Energy is transferred** and the medium provides a route by which this energy is transferred.



Energy Transport:

At any point in the **elastic medium** through which a periodic wave passes we observe a sinusoidal oscillation in time as the wave **crests** and wave **troughs** move through this region: Consider the wave below moving at **v = 1 m/s:**



At t = 0, 4, 8 seconds the medium is in equilibrium and at t = -2, 2, 6, 10 seconds there is maximum displacement corresponding to the **wave amplitude A**.

The **period T** of this wave is **8 s**. This is the time for one cycle.

The **frequency** of this wave is **f = 1 / T.** This is number of cycles per second.

The **velocity** of the wave is related to the frequency and wavelength: **V = f*λ**

Since velocity is 1 m/s, the **wavelength λ = V/f** is 8 meters. This is the distance between crests of the wave or between similar points within the period of the wave.

Don't be confused here, the horizontal axis in this figure is time and this sine wave shows the **transverse displacement** at one location in the medium. Wavelength is drawn here for reference but is a **longitudinal distance** in meters.

We have seen that the mechanical energy for the mass-spring oscillator is:

$$E = K(t) + U(t) = \frac{1}{2} k x_m{}^2 \{Sin^2(\omega t + \phi) + Cos^2(\omega t + \phi)\} = \frac{1}{2} k x_m{}^2$$

Where $\pm$ **x_m** is the amplitude of the oscillation.

Since the particles of the medium move in SHM, the energy transported by a SHM wave is likewise found to be proportional to the square of the wave amplitude.

The wave **intensity** is power transported across a unit area perpendicular to the direction of energy flow:

In **one dimension**, such as a transverse wave moving along a rope, the unit area is fixed in time [cross-section of the rope] and both the intensity of the wave and its amplitude are constant neglecting losses.

For a point source in a **3-D isotropic medium**, waves propagate outward spherically and:

$$I = \frac{P}{4\pi r^2}$$

Where **r** is distance from the source. This falls off as **$1/r^2$**

With power = energy / time**, intensity is proportional to the square of amplitude A^2**:

$$I \propto A^2$$

Further **intensity is also proportional to $1/r^2$**, such that

$$\frac{1}{r^2} \propto A^2$$

Amplitude falls off as **$1/r$** away from the source. [No losses]

Now from the mechanical energy of the wave $E = \frac{1}{2}kA^2$, we can replace **k** with

$\omega^2 m = 4\pi^2 f^2 m$ giving $E = \frac{1}{2}4\pi^2 f^2 m A^2$

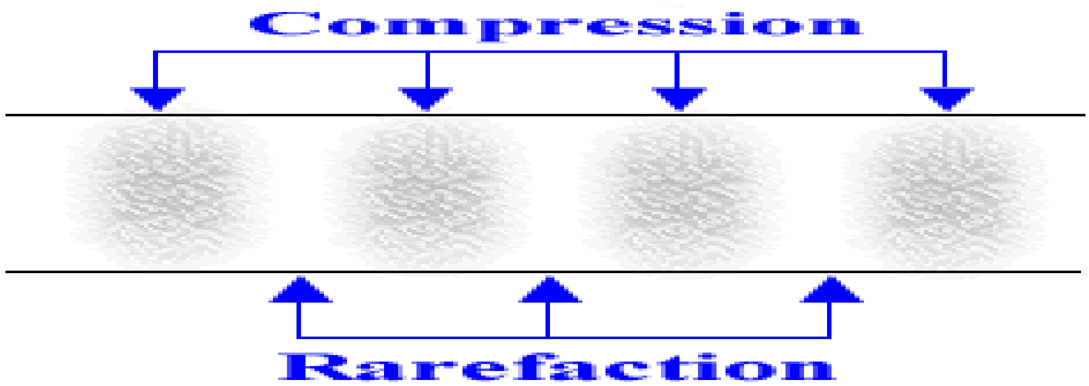
This then implies that intensity is proportional to f^2: $I \propto f^2$

Longitudinal Waves:

Longitudinal waves displace particles of the medium in a direction collinear with the direction of propagation. Springs and sound waves are two types of longitudinal waves; here we will focus on the latter.

As a sound wave propagates it creates **compressions and rarefactions** in the medium. These correspond to the crests and troughs of the **transverse wave**.

Graphically, the medium appears to periodically vary in molecular density or pressure as the molecules oscillate longitudinally:



Wavelength for a longitudinal wave is the distance between successive peaks or **over-pressures**.

Frequency corresponds to the number of compressions passing a given point per second.

Velocity is: $\mathbf{V = f*\lambda}$

Wave Velocity and Medium

The wave velocity for either transverse waves or longitudinal waves depends on the medium in which the wave propagates.

For a transverse wave along a string, the velocity is: $v = \sqrt{\dfrac{F_T}{m / L}}$ The numerator is tension, and the denominator string mass per length.

In the case of longitudinal waves in a solid,

$v = \sqrt{\dfrac{E}{\rho}}$ Where **E** is the Young's modulus of the medium and ρ its density.

In the case of longitudinal waves in a liquid or gas,

$v = \sqrt{\dfrac{B}{\rho}}$ Where **B** is the bulk modulus of the medium and ρ its density.

In each case, there is an elastic factor in the numerator and an inertial factor in the denominator.

Mechanical Wave Phenomena

When mechanical waves encounter boundaries, obstacles or other waves traveling in the same space at the same time, several outcomes are possible. Some of these include:

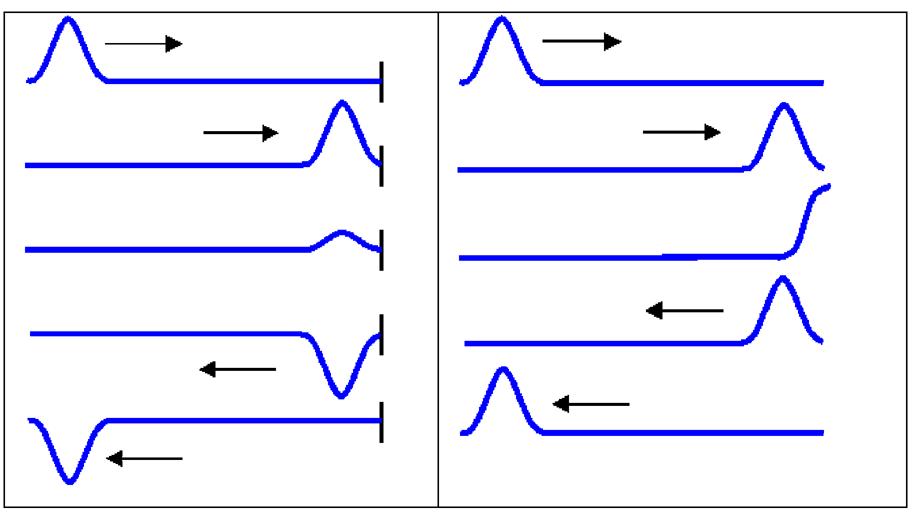
1) **Reflection: partial or total.**
2) **Interference: constructive and/or destructive**
3) **Resonance**
4) **Refraction**
5) **Diffraction**

Reflection

Reflection from a boundary or obstacle depends on the degree to which that boundary or obstacle is rigid or free to recoil.

In one-dimension, wave **reflection from a rigid boundary inverts** the outgoing wave, which then travels in the opposite direction with a velocity and amplitude equal to its incident velocity and amplitude. [**Apply Newton's 3rd Law at the wall**.]

If the boundary is free to move, then the **reflected wave is not inverted** and travels in the opposite direction with velocity and amplitude equal to that of the incident wave velocity and amplitude. [Nonzero motion at the boundary generates the reflected wave]



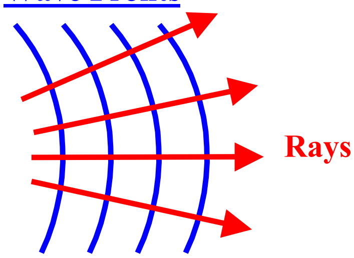
Finally, if a wave encounters a discontinuity in the propagating medium such as a variation in **mass per unit length** of a rope then there will be some amount of reflection (inverted type) and some transmission.

Reflection of two and three-dimensional mechanical waves take place according to:

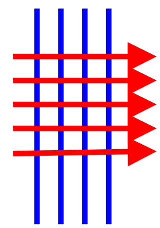
Law of Reflection:

A **wave front** corresponds to the line, circle or sphere formed by connecting the in-time crests in a propagating wave. **Rays** are vectors perpendicular to wave fronts indicating the direction of propagation.

Wave Fronts

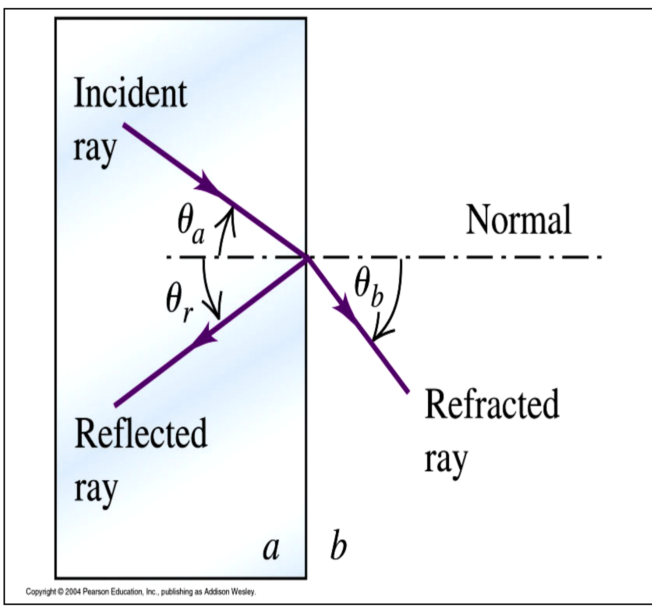


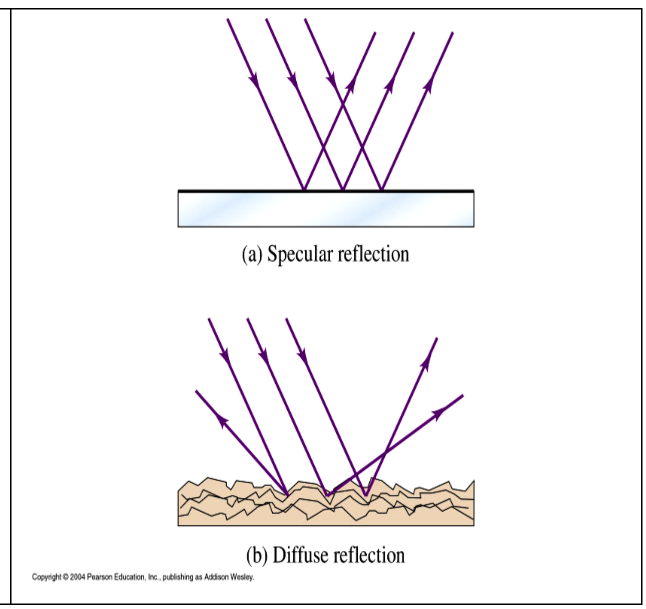
Plane Waves



At sufficiently large distances from the source or by considering a small portion of a spherical wave front, propagating waves may be approximated as **plane waves**.

The Law of Reflection is: the angle of reflection = the angle of incidence $\theta_a = \theta_r$



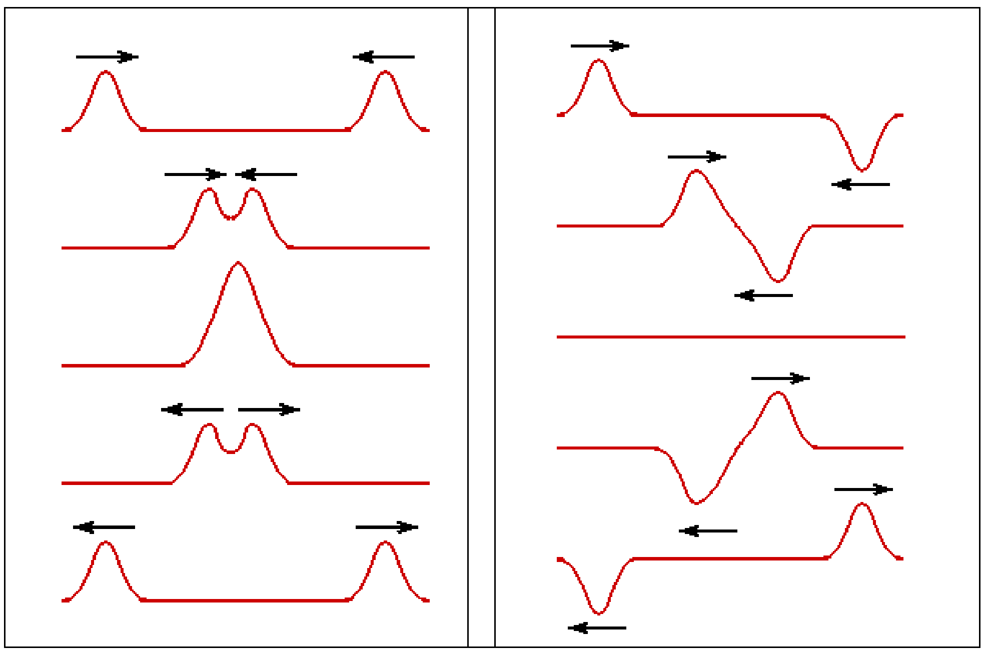


Interference

Two or more waves traveling in the same physical space at the same time may interfere with one another through **superposition**. Superposition states that the result of two or more interfering waves is the algebraic sum of the waves involved.

Constructive Interference: Waves add to result in a wave with amplitude larger than any of the incident wave amplitudes.

Destructive Interference: The algebraic sum of the incident waves results in a wave with amplitude less than one or more of the incident wave amplitudes.



From the above pictures, the importance of **phase** in the process is clear**:**

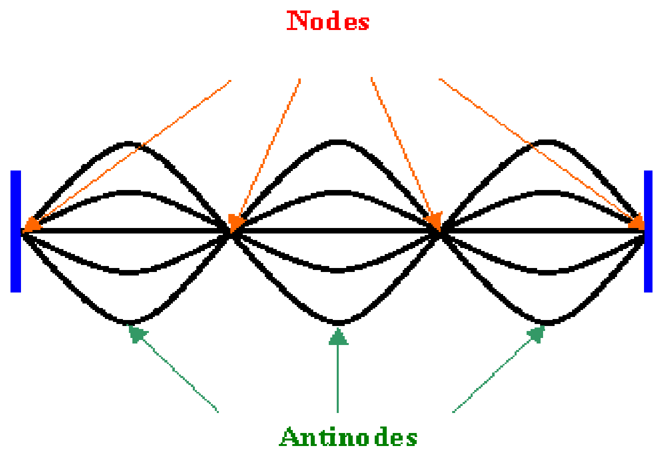
If the two waves are "in phase" as they intersect, then their peaks coincide at that point and the waves constructively interfere.

If, on the other hand, a 180^0 phase difference exist, then as the crest of wave A enters the region, a trough from wave B enters the same region and the algebraic sum goes to zero in a **totally destructive interference**.

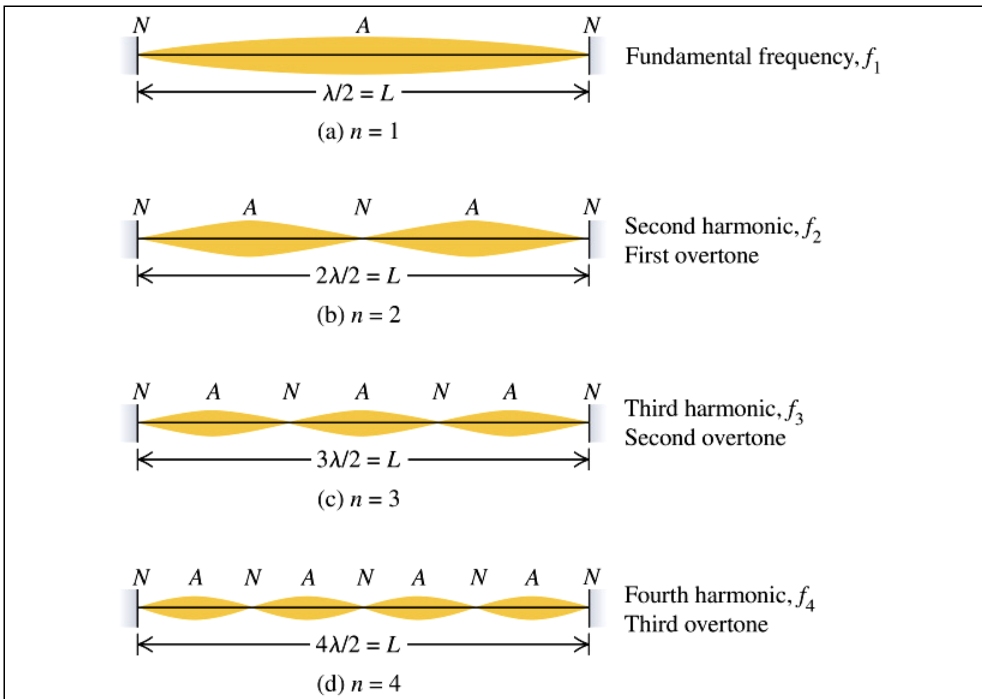
Resonance and Standing Waves

Given an oscillating source driving a one-dimensional wave that is bounded between two fixed endpoints, then at particular **resonant frequencies** the reflected waves from one end interfere with incident waves to produce a **standing wave pattern**.

Standing waves consist of points of totally destructive interference called **nodes**, and points of constructive interference known as **antinodes** ➔ maximum amplitudes.



The lowest driving frequency leading to resonance is termed the **fundamental or 1st harmonic** corresponding to the case where one half-integer wavelength oscillates within the boundary separation distance **L**.



$$L = \frac{1}{2}\lambda_1$$

$$L = \frac{2}{2}\lambda_1$$

$$L = \frac{3}{2}\lambda_1$$

$$L = \frac{4}{2}\lambda_1$$

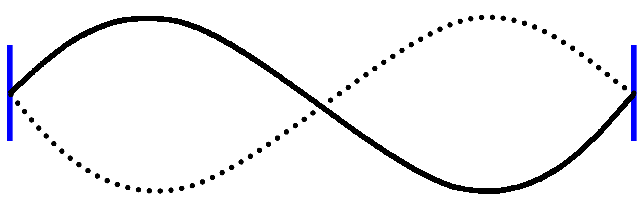
The frequency of the fundamental is $f_1 = \frac{v}{\lambda_1}$ where v is the velocity. For a string,

$$v = \sqrt{\frac{F_T}{\frac{m}{L}}}$$

Additional harmonics and resonant frequencies are possible as more half-integer wavelengths are 'fit' in between the boundary walls:

The 2nd harmonic is $f_2 = \frac{v}{\lambda_2}$. Using $L = n\frac{1}{2}\lambda_n$ determines λ_2.

In the case of n = 2, a full wavelength oscillates between the boundaries:



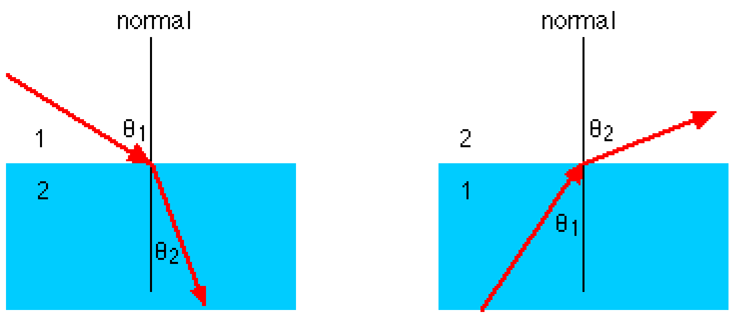
The general results for resonant frequencies are:

The **nth** harmonic has wavelength determined by the condition $L = n\frac{1}{2}\lambda_n$ and the corresponding frequencies are $f_n = \frac{v}{\lambda_n} = nf_1$

Refraction & Diffraction

Waves incident at the boundary between two different mediums will in general be partially reflected and partially transmitted. The reflected wave obeys the law of refection, and the transmitted wave is **refracted**.

Refraction of waves at a boundary interface results in a change in the incident wave propagation direction. The amount of refraction and the direction towards which it bends with respect to a **normal** at the interface depends on the speed with which the transmitted wave travels:



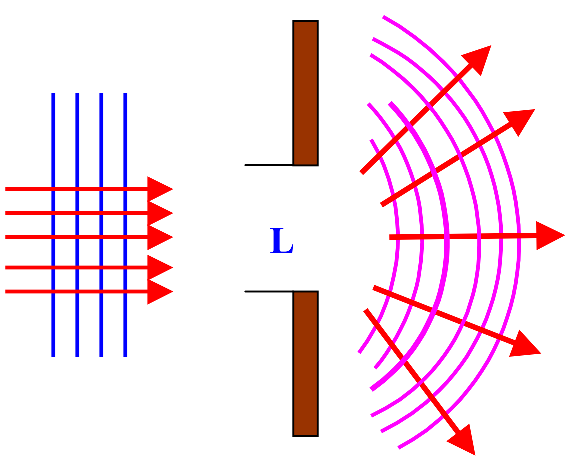
Medium 2 is more viscous and the angles are related: $\dfrac{Sin(\theta_1)}{v_1} = \dfrac{Sin(\theta_2)}{v_2}$

Diffraction

If incident wavelength is much greater than an obstacles dimension, then diffraction effects are minimal and waves pass by without much change.

When obstacle dimensions are comparable to the wavelength, then the wave bends or diffracts as it encounters an obstacle. The amount of '**shadow region**' behind the obstacle increases as the incident wavelength decreases relative to the obstacle size.

Diffraction effects also take place for waves incident on a slit:



Angular spread is $\theta_{Radians} \approx \dfrac{\lambda}{L}$